International Journal of Computer Information Systems and Industrial Management Applications.
ISSN 2150-7988 Volume 4 (2012) pp. 437-–444

Partial Approximative Set Theory: A Generalization of the Rough Set Theory

Zoltán Csajbók[1] **and Tamás Mihálydeák**[2]

[1]Department of Health Informatics, Faculty of Health, University of Debrecen,
Sóstói út 2-4, Nyíregyháza, H-4400, Hungary
csajbok.zoltan@foh.unideb.hu

[2]Department of Computer Science, Faculty of Informatics, University of Debrecen,
Egyetem tér 1, Debrecen, H-4032, Hungary
mihalydeak.tamas@inf.unideb.hu

Abstract: **The paper presents a generalization of the classical rough set theory, called the *partial approximative set theory (PAST)*. According to Pawlak's rough set theory, the *vagueness* of a subset of a finite universe U is defined by the difference of its upper and lower approximations with respect to a σ-algebra generated by an equivalence relation on U. There are two most natural ways of the generalization of this idea. In particular, the equivalence relation is replaced by either any other type of binary relations on U or an arbitrary covering of U. In this paper, our starting point will be a *partial covering* of an arbitrary universe. In general, the family of sets neither covers the universe nor forms a σ-algebra. We will put our discussions into an overall treatment called the general set theoretic approximation framework. We will investigate under what conditions our generalized upper and lower approximation pair forms Galois connection.**

Keywords: vagueness, approximation of sets, rough set theory, partial approximative set theory, Galois-connection.

I. Introduction

The rough set theory (RST) was invented by the Polish mathematician, Zdzisław Pawlak in the early 1980s [1, 2]. It can be seen as a new mathematical approach to manage uncertain, incomplete, inexact or vague knowledge [3].

In its classical form, the starting point is a nonempty *finite* set U of distinguishable objects, called the *universe of discourse*, and an *equivalence relation* ε on U. The partition of U generated by ε is denoted by U/ε, and its elements are called *ε-elementary* sets. An ε-elementary set can be viewed as a set of indiscernible objects characterized by the same available information about them [4, 5]. In addition, any union of ε-elementary sets is referred to as *definable* set.

Any subset $X \subseteq U$ can be naturally approximated by two sets called the lower and upper ε-approximations of X. The *lower ε-approximation* of X is the union of all the ε-elementary sets which are the subsets of X, whereas the *upper ε-approximation* of X is the union of all the ε-elementary sets that have a nonempty intersection with X.

The difference of upper and lower ε-approximations is called the *ε-boundary* of X. X is exact (*ε-crisp*), if its ε-boundary is the empty set, inexact (*ε-rough*) otherwise.

Let $\sigma(U/\varepsilon) \subseteq 2^U$ denote the extension of U/ε with all the unions of some ε-elementary sets and the empty set. $\sigma(U/\varepsilon)$ is a σ-algebra with the basis U/ε. In other words, $(U, \sigma(U/\varepsilon))$ is an Alexandrov topological space, where $\sigma(U/\varepsilon)$ is the family of all open and closed sets [6, 7].

In Pawlak's theory, lower and upper ε-approximations can be defined in three equivalent forms. They are based on elements of U, ε-elementary sets and the σ-algebra $\sigma(U/\varepsilon)$ [8, 9, 10]. The generalization of Pawlak's approximations can go along one of the three equivalent definitions.

The most natural generalization of Pawlak's idea is that the equivalence relation is replaced by any other type of binary relation on U [11, 6, 12]. Another way is that the partition is replaced by any covering of U [13, 14]. The third way is to use two different subsystems of the powerset of U [15]. A subsystem for the lower approximation which must be closed under unions and another for the upper approximation which, in turn, must be closed under intersections

In this paper, our starting point will be a *partial covering* of an arbitrary universe U. The family of sets generally neither covers the universe nor forms a σ-algebra. We will put our discussions into an overall treatment called the general approximation framework. Within this framework, our concepts of lower and upper approximations are straightforward *point-free* generalizations of Pawlak's ones. This new approach is called the *partial approximative set theory (PAST)*.

The rest of the paper is organized as follows. In Section II we summarize the basic notations. Section III outlines two general approximation frameworks, a large-scaled one and a much finer one. This is one of the major contribution of our paper. Section IV presents the fundamental concepts and their properties of the classic Pawlak's rough set theory. Only those facts will be considered which are important to the forthcoming ones. We provide new elementary point-free proofs for some of them. Section V sums up the basic principles of the partial approximative set theory. In Section VI we will investigate under what conditions our generalized upper and lower approximation pair forms Galois connection.

II. Basic Notations

Let U be a nonempty set. Let $\mathfrak{A} \subseteq 2^U$ be a family of subsets of U. The union of $\mathfrak{A}$ is $\bigcup \mathfrak{A} = \{x \mid \exists A \in \mathfrak{A}(x \in A)\}$, and the intersection of $\mathfrak{A}$ is $\bigcap \mathfrak{A} = \{x \mid \forall A \in \mathfrak{A}(x \in A)\}$. If $\mathfrak{A}$ is an empty family of sets we define $\bigcup \emptyset = \emptyset$ and $\bigcap \emptyset = U$.

If ϵ is an arbitrary binary relation on U, let $[x]_\epsilon$ denote the ϵ-related elements to x, i.e., $[x]_\epsilon = \{y \in U \mid (x, y) \in \epsilon\}$. The family of $[x]_\epsilon$ is denoted by X/ϵ.

A nonempty set P together with a partial order $\leq$ on P is called a *poset*, in symbols $(P, \leq)$.

A self-map $f : P \to P$ is *extensive* if $x \leq f(x)$, *contractive* if $f(x) \leq x$.

If $(P, \leq_P)$ and $(Q, \leq_Q)$ are two posets, a map $f : P \to Q$ is *monotone* when $x \leq_P y \Rightarrow f(x) \leq_Q f(y)$, and *antitone* when $x \leq_P y \Rightarrow f(y) \leq_Q f(x)$, and *order isomorphism* if f is a bijection and both f and f^{-1} are monotone.

Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two posets, and (f, g) denote a pair of maps $f : P \to Q$, $g : Q \to P$. (f, g) is a *Galois connection* between P and Q if

$$\forall p \in P\, \forall q \in Q\ (f(p) \leq_Q q \Leftrightarrow p \leq_P g(q)).$$

f is called the *lower adjoint* and g the *upper adjoint* of the Galois connection.

We also write (P, f, g, Q) for a whole Galois connection. If $P = Q$ it is said (P, f, g, P) is a Galois connection on P.

Remark 1. Here we adopted the definition of Galois connection in which the maps are monotone. It is also called the monotone or covariant form. For more details, see, e.g. [16].

The following proposition gives a useful characterization of Galois connections.

Proposition 2 (E.g., [11], Lemma 79)**.** *The pair (f, g) is a Galois connection if and only if*

1. *$\forall p \in P\, (p \leq_P g(f(p)))$ and $\forall q \in Q\, (f(g(q)) \leq_Q q)$;*
2. *the maps f and g are monotone.*

III. General Approximation Frameworks

In order to be able to treat the common features of both rough set theory and partial approximative theory uniformly, we define two general approximation frameworks, a large-scaled initial one, and a much finer general set theoretic one.

A. An Initial Approximation Framework

A large-scaled general framework of the set approximation with a pair of lower and upper approximation maps has been proposed in [17]. It is based on [18] and [19]. The framework has a specific prerequisite, in particular, *the subsets of a set are approximated by the beforehand given subsets of the set itself.*

Let U be a nonempty set and $\langle l, u\rangle$ be a pair of maps

$$l, u : 2^U \to 2^U.$$

The maps l and u are, of course, intended to be the lower and upper approximations of any subset $X \subseteq U$.

In this context, the nature of an approximation pair—beyond how they relate to one another—depends on how the lower and upper approximations of subsets relate to the subsets themselves.

The most essential features of approximation pairs $\langle l, u\rangle$ of this type are specified as follows.

1. (*Monotonicity*) The maps $l, u : 2^U \to 2^U$ are monotone.
2. (*Weak approximation property.*) A pair of maps $\langle l, u\rangle$ is the *weak approximation pair* on U if they are monotone and
$$\forall X \in 2^U\ (l(X) \subseteq u(X)).$$
3. (*Strong approximation property.*) A pair of maps $\langle l, u\rangle$ is the *strong approximation pair* on U, if each subset $X \in 2^U$ is bounded by $l(X)$ and $u(X)$:
$$\forall X \in 2^U\ (l(X) \subseteq X \subseteq u(X)).$$
4. (*Approximation hypothesis.*) A pair of maps $\langle l, u\rangle$ forms *Galois connection* on $(2^U, \subseteq)$ if
$$\forall X \in 2^U\, \forall Y \in 2^U\ (l(X) \subseteq Y \Leftrightarrow X \subseteq u(Y)).$$

Remark 3. *Ad 1.* This property is a common and reasonable assumption.

Ad 2. This constraint seems to be the weakest condition for a sensible concept of approximation of subsets in U [20, 18].

Ad 4. In [19], a new hypothesis about approximation has been drawn up recently. According to this assumption, the notion of the "approximation" may be mathematically modelled by the notion of the Galois connection.

A much finer characterization of the nature of set approximations can be obtained with *further specifications concerning the set families $l(2^U)$ and $u(2^U)$*. These additional specifications will be performed in the next Subsection.

B. A General Set Theoretic Framework of Set Approximation

Let U be an arbitrary nonempty set called the *universe of discourse*.

Let $\mathfrak{D} \subseteq 2^U$ be a nonempty family of sets so that $\emptyset \in \mathfrak{D}$ and it contains at least a nonempty subset $D \in 2^U$. The members of $\mathfrak{D}$ are called *definable sets*, while the members of $2^U \setminus \mathfrak{D}$ are *undefinable*.

We want to approximate of any subset $S \in 2^U$ from "lower side" and "upper side"—no matter what they mean at this time. We have the only requirement at the highest level of abstraction that to let the lower and upper approximations of subsets S be *definable*. We look at definable sets as *tools* to approximate subsets.

The following definition, at the next level of abstraction, is about the *minimum requirements* of lower and upper approximation mappings.

Definition 4. *A pair $\langle l, u\rangle$ of maps $l, u : 2^U \to \mathfrak{D}$ is the* weak approximation pair *on U if*

*(C1) l and u are monotone (*monotonicity*);*

*(C2) $u(\emptyset) = \emptyset$ (*normality of u*);*

*(C3) if $D \in \mathfrak{D}$, then $l(D) = D$ (*granularity of $\mathfrak{D}$*);*

*(C4) if $S \in 2^U$, then $l(S) \subseteq u(S)$ (*approximation property*).*

Clearly, the maps l, u are total and many-to-one. According to the next proposition, l is surjective, but u in not necessarily surjective.

Proposition 5. *Let $l, u : 2^U \to \mathfrak{D}$ be a weak approximation pair on U.*

1. *$l(\emptyset) = \emptyset$ (normality of l);*
2. *$\forall X \in 2^U\ (l(l(X)) = l(X))$ (idempotency of l).*
3. *$S \in \mathfrak{D}$ if and only if $l(S) = S$.*
   In other words, $l(2^U) = \mathfrak{D}$, i.e. l is surjective.
4. *$u(2^U) \subseteq l(2^U) = \mathfrak{D}$.*
   In other words, u is not necessarily surjective.

Proof.

1. By definition, $\emptyset \in \mathfrak{D}$ and so $l(\emptyset) = \emptyset$ by condition (C3).
2. $l(X) \in \mathfrak{D}$ and so $l(l(X)) = l(X)$ by condition (C3).
3. ($\Rightarrow$) It is just the same as the condition (C3).
   ($\Leftarrow$) Since $l(S) \in \mathfrak{D}$, and so $l(S) = S \in \mathfrak{D}$.
4. Let $S \in u(2^U) \subseteq \mathfrak{D}$. By the condition (C3), $S = l(S) \in l(2^U)$.

$\square$

The following example shows that each condition in Definition 4 is independent of the other three.

Example 6. Let U be a nonempty set. Let us assume that there exist $B_1, B_2 (\neq \emptyset) \in 2^U$ so that neither $B_1 \subseteq B_2$ nor $B_2 \subseteq B_1$ holds, and there exists a proper superset S of B_1 (i.e. $\emptyset \neq B_1 \subsetneqq S \neq U$).

1. Let $\mathfrak{D} = \{\emptyset, B_1, B_2, B_1 \cup B_2\}$ and $l, u : 2^U \to \mathfrak{D}$ be as follows:
$$X \mapsto l(X) = \begin{cases} B_1, & \text{if } X = B_1; \\ B_2, & \text{if } X = B_2; \\ B_1 \cup B_2, & \text{if } X = B_1 \cup B_2, U; \\ \emptyset, & \text{otherwise.} \end{cases}$$
$$X \mapsto u(X) = \begin{cases} \emptyset, & \text{if } X = \emptyset; \\ B_1, & \text{if } X = B_1; \\ B_1 \cup B_2, & \text{if } X = B_1 \cup B_2, U; \\ B_2, & \text{otherwise.} \end{cases}$$
Conditions (C2), (C3) trivially hold. Let us check the condition (C4):
$$\begin{aligned} l(\emptyset) = \emptyset &\subseteq \emptyset = u(\emptyset) \\ l(B_1) = B_1 &\subseteq B_1 = u(B_1) \\ l(B_2) = B_2 &\subseteq B_2 = u(B_2) \\ l(B_1 \cup B_2) = B_1 \cup B_2 &\subseteq B_1 \cup B_2 = u(B_1 \cup B_2) \\ l(U) = B_1 \cup B_2 &\subseteq B_1 \cup B_2 = u(U) \\ l(S) = \emptyset &\subseteq B_2 = u(S) \end{aligned}$$
and if $S'(\neq \emptyset, B_1, B_2, B_1 \cup B_2, S, U) \in 2^U$, then
$$l(S') = \emptyset \subseteq B_2 = u(S').$$
That is the condition (C4) also holds. However, in the case $B_1 \subsetneqq S$
$$\begin{aligned} l(B_1) = B_1 &\not\subseteq \emptyset = l(S) \\ u(B_1) = B_1 &\not\subseteq B_2 = u(S). \end{aligned}$$
Therefore, these l and u satisfy all the four conditions except (C1).

2. Let $\mathfrak{D} = \{\emptyset, B_1, B_2, B_1 \cup B_2\}$ and $l, u : 2^U \to \mathfrak{D}$ be as follows:
$$X \mapsto l(X) = \begin{cases} \emptyset, & \text{if } X = \emptyset; \\ B_1, & \text{if } X = B_1; \\ B_2, & \text{if } X = B_2; \\ B_1 \cup B_2, & \text{otherwise.} \end{cases}$$
$X \mapsto u(X) = B_1 \cup B_2$.
Conditions (C1), (C3), (C4) hold, but $u(\emptyset) = B_1 \cup B_2$.
Therefore, these l and u satisfy all the four conditions except (C2).

3. Let $\mathfrak{D} = \{\emptyset, B_2, B_1 \cup B_2\}$ and $l, u : 2^U \to \mathfrak{D}$ be as follows:
$$X \mapsto l(X) = \begin{cases} \emptyset, & \text{if } X = \emptyset; \\ B_2, & \text{if } X = B_1; \\ B_1 \cup B_2, & \text{otherwise.} \end{cases}$$
$$X \mapsto u(X) = \begin{cases} \emptyset, & \text{if } X = \emptyset; \\ B_1 \cup B_2, & \text{otherwise.} \end{cases}$$
Conditions (C1), (C2) trivially hold. Let us check the condition (C4):
$$\begin{aligned} l(\emptyset) = \emptyset &\subseteq \emptyset = u(\emptyset) \\ l(B_1) = B_2 &\subseteq B_1 \cup B_2 = u(B_1) \end{aligned}$$
and if $S'(\neq \emptyset, B_1) \in 2^U$, then
$$l(S') = B_1 \cup B_2 \subseteq B_1 \cup B_2 = u(S').$$
That is the condition (C4) also holds. However,
$$l(B_1) = B_2 \neq B_1.$$
Therefore, these l and u satisfy all the four conditions except (C3).

4. Let $\mathfrak{D} = \{\emptyset, B_1, B_2, B_1 \cup B_2\}$ and $l, u : 2^U \to \mathfrak{D}$ be as follows:
$$X \mapsto l(X) = \begin{cases} \emptyset, & \text{if } X = \emptyset; \\ B_1, & \text{if } X = B_1; \\ B_2, & \text{if } X = B_2; \\ B_1 \cup B_2, & \text{otherwise.} \end{cases}$$
$X \mapsto u(X) = \emptyset$.
These l and u trivially satisfy all the four conditions except (C4).

Next definition classifies the lower and upper approximation pairs as how the lower and upper approximations of a subset relate to the subset itself.

Definition 7. *A pair $\langle l, u \rangle$ of maps $l, u : 2^U \to \mathfrak{D}$ is*

- *the l-*semi-strong *approximation pair on U if it is weak and if $S \in 2^U$, then $l(S) \subseteq S$ (l is* contractive*);*
- *the u-*semi-strong *approximation pair on U if it is weak and if $S \in 2^U$, then $S \subseteq u(S)$ (u is* extensive*);*
- *the* strong *approximation pair on U if it is l-semi-strong and u-semi-strong at the same time, i.e. each subset $S \in 2^U$ is bounded by $l(X)$ and $u(X)$: $\forall S \in 2^U\ (l(S) \subseteq S \subseteq u(X)$.*

If U is a nonempty set, and $\mathfrak{D} = 2^U$, it is straightforward that the pair of maps $l, u : 2^U \to \mathfrak{D}$, $X \mapsto X$ is a strong approximation pair.

The next example shows that there are approximation pairs which are neither l-semi-strong nor u-semi-strong, not l-semi-strong but u-semi-strong, l-semi-strong but not u-semi-strong.

Example 8. Let $U = \{a, b\}$ be a nonempty set.

1. Let $\mathfrak{D} = \{\emptyset, \{a\}\}$, and the maps $l, u : 2^U \to \mathfrak{D}$ be as follows:
$$X \mapsto l(X), u(X) = \begin{cases} \emptyset, & \text{if } X = \emptyset; \\ \{a\}, & \text{otherwise.} \end{cases}$$
Conditions (C1)–(C4) can easily be checked. However, for $X = \{b\}$
$$l(\{b\}) = \{a\} \not\subseteq \{b\} \not\subseteq \{a\} = u(\{b\}). \tag{1}$$
Therefore, the approximation pair $\langle l, u\rangle$ is neither l-semi-strong nor u-semi-strong.

2. Let $\mathfrak{D} = \{\emptyset, \{a\}, \{a, b\}\}$, and $l, u : 2^U \to \mathfrak{D}$ be as follows:
$$X \mapsto l(X), u(X) = \begin{cases} \emptyset, & \text{if } X = \emptyset; \\ \{a\}, & \text{if } X = \{a\}; \\ \{a, b\}, & \text{otherwise.} \end{cases}$$
Conditions (C1)–(C4) can easily be checked. Let us check that u is extensive:
   - $\emptyset \subseteq \emptyset = u(\emptyset)$;
   - $\{a\} \subseteq \{a\} = u(\{a\})$;
   - $\{b\} \subseteq \{a, b\} = u(\{b\})$;
   - $\{a, b\} \subseteq \{a, b\} = u(\{a, b\})$.

   However, in the case $X = \{b\}$,
$$l(\{b\}) = \{a, b\} \not\subseteq \{b\} \subseteq \{a, b\} = u(\{b\}). \tag{2}$$
Therefore, the approximation pair $\langle l, u\rangle$ is not l-semi-strong, but u-semi-strong.

3. Let $\mathfrak{D} = \{\emptyset, \{a\}, \{b\}\}$, and $l, u : 2^U \to \mathfrak{D}$ be as follows:
$$X \mapsto l(X), u(X) = \begin{cases} \emptyset, & \text{if } X = \emptyset; \\ \{a\}, & \text{if } X = \{a\}; \\ \{b\}, & \text{otherwise.} \end{cases}$$
Conditions (C1)–(C4) can easily be checked. Let us check that l is contractive:
   - $l(\emptyset) = \emptyset \subseteq \emptyset$;
   - $l(\{a\}) = \{a\} \subseteq \{a\}$;
   - $l(\{b\}) = \{b\} \subseteq \{b\}$;
   - $l(\{a, b\}) = \{b\} \subseteq \{a, b\}$.

   However, in the case $X = \{a, b\}$,
$$l(\{a, b\}) = \{b\} \subseteq \{a, b\} \not\subseteq \{b\} = u(\{a, b\}). \tag{3}$$
Therefore, the approximation pair $\langle l, u\rangle$ is l-semi-strong, but not u-semi-strong.

Using the preliminary notations general approximation spaces can be defined.

Definition 9. *An ordered quadruple $\langle U, \mathfrak{D}, l, u\rangle$ is the* weak/l-semi-strong/u-semi-strong/strong *generalized approximation space, if the approximation pair $\langle l, u\rangle$ is weak/l-semi-strong/u-semi-strong/strong, respectively.*

Proposition 10. *Let $\langle U, \mathfrak{D}, l, u\rangle$ be a generalized approximation space.*

1. *If $\langle U, \mathfrak{D}, l, u\rangle$ is weak, then*
   (a) *$l(U) \subseteq \bigcup \mathfrak{D}$;*
   (b) *$l(U) = \bigcup \mathfrak{D}$ if and only if $\bigcup \mathfrak{D} \in \mathfrak{D}$.*
   (c) *$u(U) \subseteq \bigcup \mathfrak{D}$.*
2. *If $\langle U, \mathfrak{D}, l, u\rangle$ is u-semi-strong, then $u(U) = \bigcup \mathfrak{D} = U$.*

Proof.

1. (a) By the definition of l, $l(U) \in \mathfrak{D}$ and so $l(U) \subseteq \bigcup \mathfrak{D}$.
   (b) ($\Rightarrow$) By the definition of l, $l(U) = \bigcup \mathfrak{D} \in \mathfrak{D}$.
   ($\Leftarrow$) Let us assume that $\bigcup \mathfrak{D} \in \mathfrak{D}$. Since $\bigcup \mathfrak{D} \subseteq U$, then by the condition (C3) and the monotonicity of l, $l(\bigcup \mathfrak{D}) = \bigcup \mathfrak{D} \subseteq l(U)$. Comparing it with (1) (a), we obtain $l(U) = \bigcup \mathfrak{D}$.
   (c) By the definitions of u, $u(U) \in \mathfrak{D}$ and so $u(U) \subseteq \bigcup \mathfrak{D}$.
2. $\langle U, \mathfrak{D}, l, u\rangle$ is weak, thus by Proposition 10 (1)/(c), $u(U) \subseteq \bigcup \mathfrak{D}$. On the other hand, since u is monotone and extensive, $\bigcup \mathfrak{D} \subseteq U$ implies $\bigcup \mathfrak{D} \subseteq u(\bigcup \mathfrak{D}) \subseteq u(U)$. Consequently, $u(U) = \bigcup \mathfrak{D}$.
   Clearly, $u(U) \subseteq U$. Since u is extensive, thus $U \subseteq u(U)$. Therefore, $u(U) = U$.

□

In generalized approximation spaces the notion of *well approximated sets* can be introduced which we call crisp sets.

Definition 11. *Let $\langle U, \mathfrak{D}, l, u\rangle$ be a weak/l-semi-strong/u-semi-strong/ strong generalized approximation space and $S \in 2^U$.*

S is a weak/l-semi-strong/u-semi-strong/strong crisp set *with respect to the given weak/l-semi-strong/u-semi-strong/strong generalized approximation space, if $l(S) = u(S)$.*

Proposition 12. *Let $\langle U, \mathfrak{D}, l, u\rangle$ be a strong generalized approximation space.*

If $S \in 2^U$ is a strong crisp set, then S is definable.

Proof. In the strong generalized approximations space $\langle U, \mathfrak{D}, l, u\rangle$, $l(S) \subseteq S \subseteq u(S)$. Since S is crisp, therefore $l(S) = S = u(S)$, and so $S \in \mathfrak{D}$ by Proposition 5, point 3. □

In general, the crisp property of a set does not imply its definability in not strong generalized approximation spaces. One can check that in Example 8, the set $\{b\}$ in the equations (1), (2), and the set $\{a, b\}$ in the equation (3) are all crisp but none of them is definable. Of course, their lower and upper approximations are definable.

IV. Basics of Rough Set Theory

The basic concepts and properties of rough set theory can be found, e.g, in [21], [11]. Here we cite only a few of them which will be important in the following. We provide new elementary point-free proofs for some of them.

Definition 13. *A pair (U, ε), where U is a finite universe of discourse and ε is an equivalence relation on U, is called* Pawlak's approximation space.

*A subset $X \subseteq U$ is ε-*definable*, if it is a union of ε-elementary sets, otherwise X is ε-*undefinable.

By definition, the empty set is considered to be an ε-definable set.

Let $\mathfrak{D}_{U/\varepsilon}$ denote the family of ε-definable subsets of U.

Remark 14. For an evolutionary survey of approximation spaces, see [22].

The following lemma is elementary, however, in the context of Pawlak's rough set theory it is an important fact. It follows from just the fact that the partition U/ε consists of nonempty pairwise disjoint subsets of U.

Lemma 15. $\forall \mathfrak{X} \in 2^{U/\varepsilon}\ \forall X \in U/\varepsilon\ (X \subseteq \bigcup \mathfrak{X} \Leftrightarrow X \in \mathfrak{X})$.

Proposition 16. *Let (U, ε) be Pawlak's approximation space.*

Then $(2^{U/\varepsilon}, \subseteq)$ and $(\mathfrak{D}_{U/\varepsilon}, \subseteq)$ are order isomorphic via the map $\mathfrak{u}_\varepsilon : 2^{U/\varepsilon} \to \mathfrak{D}_{U/\varepsilon}, \mathfrak{X} \mapsto \bigcup \mathfrak{X}$.

Proof. We show that the map $\mathfrak{u}_\varepsilon$ is a bijection and both $\mathfrak{u}_\varepsilon$ and $\mathfrak{u}_\varepsilon^{-1}$ are monotone.

Let $\mathfrak{D}_1, \mathfrak{D}_2 \in 2^{U/\varepsilon}$ be such that $\bigcup \mathfrak{D}_1 = \bigcup \mathfrak{D}_2 \in \mathfrak{D}_{U/\varepsilon}$. By Lemma 15, $\forall X \in 2^U (X \in \mathfrak{D}_1 \Leftrightarrow X \subseteq \bigcup \mathfrak{D}_1 = \bigcup \mathfrak{D}_2 \Leftrightarrow X \in \mathfrak{D}_2)$, i.e., $\mathfrak{D}_1 = \mathfrak{D}_2$, thus $\mathfrak{u}_\varepsilon$ is injective. By definition of $\mathfrak{D}_{U/\varepsilon}$, $\mathfrak{u}_\varepsilon$ is surjective. Consequently, $\mathfrak{u}_\varepsilon$ is a bijection.

Clearly, the map $\mathfrak{u}_\varepsilon$ is monotone, since $\mathfrak{X}_1, \mathfrak{X}_2 \in 2^{U/\varepsilon}$, $\mathfrak{X}_1 \subseteq \mathfrak{X}_2$ just implies $\bigcup \mathfrak{X}_1 \subseteq \bigcup \mathfrak{X}_2$.

Now, let $D_1, D_2 \in \mathfrak{D}_{U/\varepsilon}$ be so that $D_1 \subseteq D_2$. Since $\mathfrak{u}_\varepsilon$ is a bijection, there exist unique $\mathfrak{u}_\varepsilon^{-1}(D_1) = \mathfrak{X}_1, \mathfrak{u}_\varepsilon^{-1}(D_2) = \mathfrak{X}_2 \in 2^U$ so that $D_1 = \bigcup \mathfrak{X}_1, D_2 = \bigcup \mathfrak{X}_2$. By Lemma 15, $\forall X \in 2^U (X \in \mathfrak{X}_1 \Leftrightarrow X \subseteq \bigcup \mathfrak{X}_1 \subseteq \bigcup \mathfrak{X}_2 \Leftrightarrow X \in \mathfrak{X}_2)$, i.e., $\mathfrak{X}_1 \subseteq \mathfrak{X}_2$, and so $\mathfrak{u}_\varepsilon^{-1}$ is also monotone. □

In Pawlak's approximation spaces, the lower and upper approximations of X can be defined in the *point-free* manner—based on the ε-elementary sets, and in the *point-wise* manner—based on the elements.

Definition 17. *Let (U, ε) be Pawlak's approximation space, and $X \in 2^U$ be a subset of U.*

The lower ε-approximation *of X is*

$$\begin{aligned} \underline{\varepsilon}(X) &= \bigcup \{Y \mid Y \in U/\varepsilon, Y \subseteq X\} \\ &= \{x \in U \mid [x]_\varepsilon \subseteq X\}, \end{aligned}$$

and the upper ε-approximation *of X is*

$$\begin{aligned} \overline{\varepsilon}(X) &= \bigcup \{Y \mid Y \in U/\varepsilon, Y \cap X \neq \emptyset\} \\ &= \{x \in U \mid [x]_\varepsilon \cap X \neq \emptyset\}. \end{aligned}$$

*The set $B_\varepsilon(X) = \overline{\varepsilon}(X) \setminus \underline{\varepsilon}(X)$ is the ε-*boundary *of X.*

*X is ε-*crisp*, if $B_\varepsilon(X) = \emptyset$, otherwise X is ε-*rough.

It follows from just the definitions that $\underline{\varepsilon}(X), \overline{\varepsilon}(X) \in \mathfrak{D}_{U/\varepsilon}$, the maps $\underline{\varepsilon}, \overline{\varepsilon} : 2^U \to \mathfrak{D}_{U/\varepsilon}$ are total, onto and many-to-one.

Proposition 18 ([21], Proposition 2.1, point a))**.** *Let (U, ε) be Pawlak's approximation space. Then $X \in \mathfrak{D}_{U/\varepsilon}$ if and only if $\underline{\varepsilon}(X) = \overline{\varepsilon}(X)$.*

Proposition 19 ([21], Proposition 2.2, points 1)**.** *Let (U, ε) be Pawlak's approximation space. Then*

$$\forall X \in 2^U (\underline{\varepsilon}(X) \subseteq X \subseteq \overline{\varepsilon}(X)),$$

that is, the maps $\underline{\varepsilon}$ and $\overline{\varepsilon}$ are contractive and extensive, respectively.

In other words, the pair of maps $\underline{\varepsilon}(X)$ and $\overline{\varepsilon}(X)$ is a strong approximation pair on U.

Corollary 20. *$\underline{\varepsilon}(X) = X$ if and only if $X = \overline{\varepsilon}(X)$.*

Proof. Since $\underline{\varepsilon}(X) \in \mathfrak{D}_{U/\varepsilon}$ ($\overline{\varepsilon}(X) \in \mathfrak{D}_{U/\varepsilon}$), then $X = \underline{\varepsilon}(X) \in \mathfrak{D}_{U/\varepsilon}$ ($X = \overline{\varepsilon}(X) \in \mathfrak{D}_{U/\varepsilon}$), and so, by Proposition 18, $X = \underline{\varepsilon}(X) = \overline{\varepsilon}(X)$ ($X = \overline{\varepsilon}(X) = \underline{\varepsilon}(X)$). □

Proposition 21. *Let (U, ε) be Pawlak's approximation space and $X \subseteq U$.*

1. *X is ε-crisp if and only if X is ε-definable.*
2. *X is ε-rough if and only if X is ε-undefinable.*

Proof.

1. ($\Rightarrow$) X is ε-crisp $\Leftrightarrow B_\varepsilon(X) = \overline{\varepsilon}(X) \setminus \underline{\varepsilon}(X) = \emptyset \Leftrightarrow \overline{\varepsilon}(X) \subseteq \underline{\varepsilon}(X)$. Proposition 19 implies $\underline{\varepsilon}(X) \subseteq \overline{\varepsilon}(X)$, and so $\underline{\varepsilon}(X) = \overline{\varepsilon}(X)$. According to Proposition 18, $\underline{\varepsilon}(X) = \overline{\varepsilon}(X) \Leftrightarrow X \in \mathfrak{D}_{U/\varepsilon}$.

   ($\Leftarrow$) Since $X \in \mathfrak{D}_{U/\varepsilon} \Leftrightarrow \underline{\varepsilon}(X) = \overline{\varepsilon}(X)$, so $B_\varepsilon(X) = \overline{\varepsilon}(X) \setminus \underline{\varepsilon}(X) = \emptyset$ trivially satisfies.
2. It is the contrapositive version of 1.

□

As a consequence of Proposition 21, the notions 'ε-crisp' and 'ε-definable' are synonymous to each other, and so are 'ε-rough' and 'ε-undefinable'.

Lower and upper ε-approximations can be generalized via their *point-wise* definitions [11].

Definition 22. *Let ϵ be an arbitrary binary relation on U and $X \in 2^U$. The* lower ϵ-approximation *of X is*

$$\underline{\epsilon}(X) = \{x \in U \mid [x]_\epsilon \subseteq X\},$$

and the upper ϵ-approximation *of X is*

$$\overline{\epsilon}(X) = \{x \in U \mid [x]_\epsilon \cap X \neq \emptyset\}.$$

If ϵ^{-1} denotes the inverse relation of ϵ, in the same manner one can also define lower and upper ϵ^{-1}-approximations.

Proposition 23 ([11], Proposition 134)**.** *Let ϵ be an arbitrary binary relation on U.*

Then $(2^U, \overline{\epsilon}, \underline{\epsilon^{-1}}, 2^U)$ and $(2^U, \overline{\epsilon^{-1}}, \underline{\epsilon}, 2^U)$ are Galois connections on $(2^U, \subseteq)$.

Some properties of lower and upper ϵ-approximations are expressed by properties of binary relations and vice versa.

Proposition 24. *Let ϵ be an arbitrary binary relation on U.*

1. *The pair $(\underline{\epsilon}, \overline{\epsilon})$ is a weak approximation pair if and only if ϵ is connected.*
2. *The pair $(\underline{\epsilon}, \overline{\epsilon})$ is a strong approximation pair if and only if ϵ is reflexive.*

3. *The pair $(\overline{\epsilon}, \underline{\epsilon})$ is a Galois connection on $(2^U, \subseteq)$ if and only if ϵ is symmetric.*

   In particular, if ε is an equivalence relation on U, then $(2^U, \overline{\varepsilon}, \underline{\varepsilon}, 2^U)$ is a Galois connection on 2^U.

Proof. In [11]. 1. Proposition 136.; 2. Proposition 137.; 3. Proposition 138. □

It can be shown that even if the relation ϵ is symmetric, it is not sufficient that the lower and upper ϵ-approximations defined in the *point-free* manner form a Galois connection ([23], Example 3.10).

V. Partial Approximation of Sets

In practice, there are attributes which do not characterize all members of an observed collection of objects.

Some illustrative examples:

- With the property 'color of hair' bald men cannot be characterized.
- An infinite set is investigated via a finite family of its finite subsets. For instance, a number theorist studies the regularities of natural numbers using computers.
- Security policies are partial-natured in corporate information security management. Typically some policies may only apply to specific hardware appliances, software applications or type of information.

Moreover, there are some features with which a set and its *complement* cannot be described at the same time. For instance, complements of recursively enumerable sets are not necessarily recursively enumerable. The membership of recursively enumerable sets can effectively be determined by a *finite amount* of information, while the determination of their nonmembership requires an *infinite amount* of information [24]. That is, the complement of a recursively enumerable set cannot necessarily be determined effectively. In other words, the recursively enumerable sets can be managed by computers (e.g., via a special rewriting system, the Markov algorithm [25]). Thus, this is an important *practical* partial approximation problem: how can we approximate an arbitrary set with recursively enumerable sets?

Throughout this section let U be a nonempty set called the *universe of discourse*.

A. Definable Sets

The first definition gives us the fundamental sets of our framework which can be considered as our *primary tools*.

Definition 25. *Let $\mathfrak{B} = \{B_i \mid i \in I\} \subseteq 2^U$ be a nonempty family of nonempty subsets of U, where I denotes an index set.*

$\mathfrak{B}$ *is called the* base system, *its elements are the* $\mathfrak{B}$-sets.

Some extensions of the base set $\mathfrak{B}$ can be defined by means of $\mathfrak{B}$. It can be seen as *derived tools*. The next definition is about a possible extension of $\mathfrak{B}$.

Definition 26. *A set family $\mathfrak{S} \subseteq 2^U$ is* $\mathfrak{B}$-definable *if its elements are $\mathfrak{B}$-sets, otherwise $\mathfrak{S}$ is* $\mathfrak{B}$-undefinable.

A nonempty subset $S \in 2^U$ is $\mathfrak{B}$-definable *if there exists a $\mathfrak{B}$-definable set family $\mathfrak{S} = \{B_i \mid i \in I_S \subseteq I, B_i \in \mathfrak{B}\}$ so that $S = \bigcup \mathfrak{S}$, otherwise S is* $\mathfrak{B}$-undefinable.

The empty set is considered to be a $\mathfrak{B}$-definable set.

Let $\mathfrak{D}_{\mathfrak{B}}$ denote the family of $\mathfrak{B}$-definable subsets of U.

Notice that $\emptyset \in \mathfrak{D}_{\mathfrak{B}}$ and $\mathfrak{B} \subseteq \mathfrak{D}_{\mathfrak{B}}$, therefore $\mathfrak{D}_{\mathfrak{B}}$ contains at least a nonempty subset of U and $\bigcup \mathfrak{B} = \bigcup \mathfrak{D}_{\mathfrak{B}} \in \mathfrak{D}_{\mathfrak{B}}$. It is straightforward that $\mathfrak{D}_{\mathfrak{B}}$ does not form a σ-algebra at all.

We will need the following notion.

Definition 27. *The base system $\mathfrak{B} \subseteq 2^U$ is* single-layered, *if*

$$\forall B \in \mathfrak{B}\ \forall \mathfrak{B}' \subseteq \mathfrak{B} \setminus \{B\}\, (B \cap \bigcup \mathfrak{B}' \neq B).$$

Informally, a base system $\mathfrak{B}$ is single-layered if every primary and derived tools has at least one element which can be characterized by exactly one primary tool.

Some properties of rough set theory can be preserved in some wise with the notion of single-layered. For instance:

Proposition 28 ([23], Proposition 4.5; analogous with Proposition 16)**.** *Let $\mathfrak{B} \subseteq 2^U$ be a base system. Then $(2^{\mathfrak{B}}, \subseteq)$ and $(\mathfrak{D}_{\mathfrak{B}}, \subseteq)$ are order isomorphic via the map $\mathfrak{u}_{\mathfrak{B}} : 2^{\mathfrak{B}} \to \mathfrak{D}_{\mathfrak{B}}, \mathfrak{X} \mapsto \bigcup \mathfrak{X}$ if and only if $\mathfrak{B}$ is single-layered.*

B. Lower and Upper $\mathfrak{B}$-Approximations

Definition 29. *Let $\mathfrak{B} \subseteq 2^U$ be a base system and X be a subset of U.*

The lower $\mathfrak{B}$-approximation *of X is*

$$\mathfrak{C}^{\flat}_{\mathfrak{B}}(X) = \bigcup\{Y \mid Y \in \mathfrak{B}, Y \subseteq X\}, \tag{4}$$

and the upper $\mathfrak{B}$-approximation *of X is*

$$\mathfrak{C}^{\sharp}_{\mathfrak{B}}(X) = \bigcup\{Y \mid Y \in \mathfrak{B}, Y \cap X \neq \emptyset\}. \tag{5}$$

Notice that $\mathfrak{C}^{\flat}_{\mathfrak{B}}$ and $\mathfrak{C}^{\sharp}_{\mathfrak{B}}$ are straightforward *point-free* generalizations of Pawlak's lower and upper ε-approximations.

Clearly, $\mathfrak{C}^{\flat}_{\mathfrak{B}}(X), \mathfrak{C}^{\sharp}_{\mathfrak{B}}(X) \in \mathfrak{D}_{\mathfrak{B}}$, and the maps $\mathfrak{C}^{\flat}_{\mathfrak{B}}, \mathfrak{C}^{\sharp}_{\mathfrak{B}} : 2^U \to \mathfrak{D}_{\mathfrak{B}}$ are total, onto and many-to-one.

Proposition 30. *Let $\mathfrak{B} \subseteq 2^U$ be a base system.*

1. *$\forall S \in 2^U$ $(\mathfrak{C}^{\flat}_{\mathfrak{B}}(S) \subseteq S)$—that is $\mathfrak{C}^{\flat}_{\mathfrak{B}}$ is contractive.*
2. *$\forall S \in 2^U$ $(S \subseteq \mathfrak{C}^{\sharp}_{\mathfrak{B}}(S))$ if and only if $\bigcup \mathfrak{B} = U$—that is $\mathfrak{C}^{\sharp}_{\mathfrak{B}}$ is extensive if and only if $\mathfrak{B}$ covers the universe.*

Proof. 1. is straightforward.

2. $(\Rightarrow)$ $U \subseteq \mathfrak{C}^{\sharp}_{\mathfrak{B}}(U) = \bigcup\{B \mid B \in \mathfrak{B}, B \subseteq U\} = \bigcup \mathfrak{B}$. Of course, $\bigcup \mathfrak{B} \subseteq U$, and so $\bigcup \mathfrak{B} = U$.

$(\Leftarrow)$ $\forall S \in 2^U (S \subseteq U = \bigcup \mathfrak{B})$, thus we get

$$\begin{aligned} S &\subseteq \bigcup(\mathfrak{B} \setminus \{B \mid B \in \mathfrak{B}, B \cap S = \emptyset\}) \\ &= \bigcup\{B \mid B \in \mathfrak{B}, B \cap S \neq \emptyset\} = \mathfrak{C}^{\sharp}_{\mathfrak{B}}(S). \end{aligned}$$

□

Proposition 31. *Let $\mathfrak{B} \subseteq 2^U$ be a base system.*

1. *The maps $\mathfrak{C}^{\flat}_{\mathfrak{B}}, \mathfrak{C}^{\sharp}_{\mathfrak{B}} : 2^U \to \mathfrak{D}_{\mathfrak{B}}$ are monotone.*
2. *$\mathfrak{C}^{\sharp}_{\mathfrak{B}}(\emptyset) = \emptyset$.*
3. *If $D \in \mathfrak{D}_{\mathfrak{B}}$, then $\mathfrak{C}^{\flat}_{\mathfrak{B}}(D) = D$.*
4. *If $S \in 2^U$, then $\mathfrak{C}^{\flat}_{\mathfrak{B}}(S) \subseteq \mathfrak{C}^{\sharp}_{\mathfrak{B}}(S)$.*

Proof. 1., 2. and 4. are straightforward by the definition of lower and upper $\mathfrak{B}$-approximations.

3. Clearly, if $\emptyset \in \mathfrak{D}_{\mathfrak{B}}$, then $\mathfrak{C}^{\flat}_{\mathfrak{B}}(\emptyset) = \emptyset$.

If $\emptyset \neq D \in \mathfrak{D}_{\mathfrak{B}}$, there exists at least one nonempty family of sets $\mathfrak{B}' \subseteq \mathfrak{B}$ so that $D = \bigcup \mathfrak{B}' = \bigcup\{B \mid B \in \mathfrak{B}', B \subseteq D\} \subseteq \bigcup\{B \mid B \in \mathfrak{B}, B \subseteq D\} = \mathfrak{C}^{\flat}_{\mathfrak{B}}(D)$. On the other hand, we have $\mathfrak{C}^{\flat}_{\mathfrak{B}}(D) \subseteq D$. Thus $\mathfrak{C}^{\flat}_{\mathfrak{B}}(D) = D$. □

In the language of general set theoretic framework of set approximation the pair $\langle \mathfrak{C}^{\flat}_{\mathfrak{B}}, \mathfrak{C}^{\sharp}_{\mathfrak{B}} \rangle$ of maps $\mathfrak{C}^{\flat}_{\mathfrak{B}}, \mathfrak{C}^{\sharp}_{\mathfrak{B}} : 2^U \to \mathfrak{D}_{\mathfrak{B}}$ is an l-semi-strong approximation pair, and it is a strong one if and only if the base system $\mathfrak{B}$ covers the universe.

Moreover, the so-called $\mathfrak{B}$-approximation space $\langle U, \mathfrak{D}_{\mathfrak{B}}, \mathfrak{C}^{\flat}_{\mathfrak{B}}, \mathfrak{C}^{\sharp}_{\mathfrak{B}} \rangle$ is l-semi-strong and it is strong if and only if the base system $\mathfrak{B}$ covers the universe.

VI. Galois Connections in $\mathfrak{B}$-approximation Spaces

Let us investigate what conditions have to be satisfied by the l-semi-strong $\mathfrak{B}$-approximation space $\langle U, \mathfrak{B}, \mathfrak{C}^{\flat}_{\mathfrak{B}}, \mathfrak{C}^{\sharp}_{\mathfrak{B}} \rangle$ so that the pair $\langle \mathfrak{C}^{\sharp}_{\mathfrak{B}}, \mathfrak{C}^{\flat}_{\mathfrak{B}} \rangle$ forms a Galois connection on U.

The next proposition answers the first half of the Point 1 in Proposition 2.

Proposition 32. *Let $\langle U, \mathfrak{B}, \mathfrak{C}^{\flat}_{\mathfrak{B}}, \mathfrak{C}^{\sharp}_{\mathfrak{B}} \rangle$ be an l-semi-strong $\mathfrak{B}$-approximation space.*

Then $\forall X \in 2^U (X \subseteq \mathfrak{C}^{\flat}_{\mathfrak{B}}(\mathfrak{C}^{\sharp}_{\mathfrak{B}}(X)))$ if and only if $\bigcup \mathfrak{B} = U$.

Proof. $(\Rightarrow)$ By a contradiction, let us assume that $\bigcup \mathfrak{B} \neq U$. Accordingly, $\exists X'(\neq \emptyset) \subseteq U \setminus \bigcup \mathfrak{B}$. Hence, $\mathfrak{C}^{\flat}_{\mathfrak{B}}(\mathfrak{C}^{\sharp}_{\mathfrak{B}}(X')) = \emptyset$, which gives $\emptyset \neq X' \subseteq \mathfrak{C}^{\flat}_{\mathfrak{B}}(\mathfrak{C}^{\sharp}_{\mathfrak{B}}(X')) = \emptyset$, a contradiction.

$(\Leftarrow)$ $\mathfrak{C}^{\sharp}_{\mathfrak{B}}(X) \in \mathfrak{D}_{\mathfrak{B}}$, and so, by Proposition 31 Point 3, $\mathfrak{C}^{\flat}_{\mathfrak{B}}(\mathfrak{C}^{\sharp}_{\mathfrak{B}}(X)) = \mathfrak{C}^{\sharp}_{\mathfrak{B}}(X)$. Since $\bigcup \mathfrak{B} = U$, by Proposition 30 Point 2, $\mathfrak{C}^{\sharp}_{\mathfrak{B}}$ is extensive, thus $X \subseteq \mathfrak{C}^{\sharp}_{\mathfrak{B}}(X) = \mathfrak{C}^{\flat}_{\mathfrak{B}}(\mathfrak{C}^{\sharp}_{\mathfrak{B}}(X))$. □

Let us take up the question of the second half of the Point 1 in Proposition 2. In general, it also does not hold.

Proposition 33. *Let $\langle U, \mathfrak{B}, \mathfrak{C}^{\flat}_{\mathfrak{B}}, \mathfrak{C}^{\sharp}_{\mathfrak{B}} \rangle$ be an l-semi-strong $\mathfrak{B}$-approximation space.*

Then the base system $\mathfrak{B}$ is single-layered and $\forall X \in 2^U\ (\mathfrak{C}^{\sharp}_{\mathfrak{B}}(\mathfrak{C}^{\flat}_{\mathfrak{B}}(X)) \subseteq X)$ if and only if the $\mathfrak{B}$-sets are pairwise disjoint.

Proof. $(\Rightarrow)$ By a contradiction, let us assume that the $\mathfrak{B}$-sets are not pairwise disjoint, i.e.

$$\exists B_1, B_2 \in \mathfrak{B}\ (B_1 \neq B_2 \wedge B_1 \cap B_2 \neq \emptyset).$$

Because $\mathfrak{B}$ is single-layered, neither $B_1 \subseteq B_2$ nor $B_2 \subseteq B_1$ holds. Hence, e.g. for B_1, we have

$$\mathfrak{C}^{\sharp}_{\mathfrak{B}}(\mathfrak{C}^{\flat}_{\mathfrak{B}}(B_1)) = \mathfrak{C}^{\sharp}_{\mathfrak{B}}(B_1) = \bigcup\{Y \mid Y \in \mathfrak{B}, Y \cap B_1 \neq \emptyset\}.$$

Thus we get $B_1 \cup B_2 \subseteq \mathfrak{C}^{\sharp}_{\mathfrak{B}}(\mathfrak{C}^{\flat}_{\mathfrak{B}}(B_1))$, and so $\mathfrak{C}^{\sharp}_{\mathfrak{B}}(\mathfrak{C}^{\flat}_{\mathfrak{B}}(B_1)) \nsubseteq B_1$, a contradiction.

$(\Leftarrow)$ Clearly, if the $\mathfrak{B}$-sets are pairwise disjoint, $\mathfrak{B}$ is single-layered. Furthermore, $\mathfrak{C}^{\sharp}_{\mathfrak{B}}(\mathfrak{C}^{\flat}_{\mathfrak{B}}(\emptyset)) = \mathfrak{C}^{\sharp}_{\mathfrak{B}}(\emptyset) = \emptyset \subseteq \emptyset$ holds, independently of that the $\mathfrak{B}$-sets are pairwise disjoint or not.

Let $\emptyset \neq X \in 2^U$. If $\mathfrak{C}^{\flat}_{\mathfrak{B}}(X) = \emptyset$, then $\mathfrak{C}^{\sharp}_{\mathfrak{B}}(\emptyset) = \emptyset \subseteq X$. Let $\mathfrak{C}^{\flat}_{\mathfrak{B}}(X) = \bigcup \mathfrak{B}' \neq \emptyset$ for a family of $\mathfrak{B}$-sets $\mathfrak{B}' \subseteq \mathfrak{B}$. Because the map $\mathfrak{C}^{\flat}_{\mathfrak{B}}$ is contractive, $\mathfrak{C}^{\flat}_{\mathfrak{B}}(X) = \bigcup \mathfrak{B}' \subseteq X$. Since the $\mathfrak{B}$-sets are pairwise disjoint, $\{Y \mid Y \in \mathfrak{B}, Y \cap \bigcup \mathfrak{B}' \neq \emptyset\} = \{Y \mid Y \in \mathfrak{B}', Y \cap \bigcup \mathfrak{B}' \neq \emptyset\}$. Thus we get

$$\begin{aligned}
\mathfrak{C}^{\sharp}_{\mathfrak{B}}(\mathfrak{C}^{\flat}_{\mathfrak{B}}(X)) &= \mathfrak{C}^{\sharp}_{\mathfrak{B}}(\textstyle\bigcup \mathfrak{B}') \\
&= \textstyle\bigcup\{Y \mid Y \in \mathfrak{B}, Y \cap \bigcup \mathfrak{B}' \neq \emptyset\} \\
&= \textstyle\bigcup\{Y \mid Y \in \mathfrak{B}', Y \cap \bigcup \mathfrak{B}' \neq \emptyset\} \\
&= \textstyle\bigcup\{Y \mid Y \in \mathfrak{B}', Y \subseteq \bigcup \mathfrak{B}'\} \\
&\subseteq \textstyle\bigcup\{Y \mid Y \in \mathfrak{B}, Y \subseteq X\} \subseteq X.
\end{aligned}$$

□

Theorem 34. *Let $\langle U, \mathfrak{B}, \mathfrak{C}^{\flat}_{\mathfrak{B}}, \mathfrak{C}^{\sharp}_{\mathfrak{B}} \rangle$ be an l-semi-strong $\mathfrak{B}$-approximation space.*

The base system $\mathfrak{B}$ is single-layered and the pair $\langle \mathfrak{C}^{\sharp}_{\mathfrak{B}}, \mathfrak{C}^{\flat}_{\mathfrak{B}} \rangle$ forms a Galois connection on $(2^U, \subseteq)$ if and only if the base system $\mathfrak{B}$ is a partition of U.

Proof. The maps $\mathfrak{C}^{\sharp}_{\mathfrak{B}}$ and $\mathfrak{C}^{\flat}_{\mathfrak{B}}$ are monotone, and so by Propositions 32 and 33, the conditions in Proposition 2 hold. □

If the base system $\mathfrak{B}$ is a partition of U, then $\mathfrak{B}$ is trivially single-layered.

However, if the $\langle \mathfrak{C}^{\sharp}_{\mathfrak{B}}, \mathfrak{C}^{\flat}_{\mathfrak{B}} \rangle$ forms a Galois connection in an l-semi-strong $\mathfrak{B}$-approximation space, it can be proven that $\mathfrak{B}$ is a partition of U without the assumption that $\mathfrak{B}$ is single-layered. For details see [23], Theorem 4.14.

VII. Conclusion

In this paper, first, we have presented a general set theoretic approximation framework. Within this framework, a particular partial approximation space has been proposed. In Pawlak's space upper and lower approximations form a Galois connection on U. We have investigated what conditions have to be satisfied by our generalized upper and lower approximation pair forms a Galois connection on U.

References

[1] Z. Pawlak, "Information systems theoretical foundations," *Information Systems*, vol. 6, no. 3, 1981, pp. 205–218.

[2] ——, "Rough sets," *International Journal of Information and Computer Science*, vol. 11, no. 5, 1982, pp. 341–356.

[3] ——, "Some issues on rough sets," in *Transactions on Rough Sets I*, ser. Lecture Notes in Computer Science, J. F. Peters, A. Skowron, J. W. Grzymala-Busse, B. Kostek, R. W. Swiniarski, and M. S. Szczuka, Eds. Springer Berlin / Heidelberg, vol. 3100, pp. 1–58.

[4] Z. Pawlak and A. Skowron, "Rudiments of rough sets," *Information Sciences*, vol. 177, no. 1, 2007, pp. 3–27.

[5] A. Skowron, “Vague concepts: A rough-set approach,” in *Proceedings of EUROFUSE 2004*, B. De Baets, R. De Caluwe, G. De Tré, J. Fodor, J. Kacprzyk, and S. Zadrożny, Eds. Warszawa: Akademicka Oficyna Wydawnicza EXIT, 2004, pp. 480–493.

[6] J. Järvinen, S. Radeleczki, and L. Veres, “Rough sets determined by quasiorders,” *Order*, vol. 26, no. 4, 2009, pp. 337–355.

[7] A. Skowron, “On topology in information system,” *Bulletin of the Polish Academy of Sciences, Mathematics*, vol. 36, 1988, pp. 477–479.

[8] F. Xu, Y. Yao, and D. Miao, “Rough set approximations in formal concept analysis and knowledge spaces,” in *ISMIS*, ser. Lecture Notes in Computer Science, A. An, S. Matwin, Z. W. Ras, and D. Slezak, Eds., vol. 4994. Springer, 2008, pp. 319–328.

[9] Y. Yao and Y. Chen, “Rough set approximations in formal concept analysis,” in *Transactions on Rough Sets V*, ser. Lecture Notes in Computer Science, J. F. Peters and A. Skowron, Eds., vol. 4100. Springer, 2006, pp. 285–305.

[10] Y. Y. Yao, “On generalizing rough set theory,” in *Proceedings of the 9th International Conference Rough Sets, Fuzzy Sets, Data Mining, and Granular Computing (RSFDGrC 2003)*, ser. LNAI 2639. Springer-Verlag, 2003, pp. 44–51.

[11] J. Järvinen, “Lattice theory for rough sets,” in *Transactions on Rough Sets VI: Commemorating the Life and Work of Zdzisław Pawlak, Part I*, ser. Lecture Notes in Computer Science, J. F. Peters, A. Skowron, I. Düntsch, J. W. Grzymała-Busse, E. Orłowska, and L. Polkowski, Eds. Berlin, Heidelberg: Springer-Verlag, 2007, vol. 4374, pp. 400–498.

[12] Y. Y. Yao, “Constructive and algebraic methods of the theory of rough sets,” *Information Sciences*, vol. 109, no. 1–4, 1998, pp. 21–47.

[13] W. Zhu, “Topological approaches to covering rough sets,” *Information Sciences*, vol. 177, no. 6, 2007, pp. 1499–1508.

[14] ——, “Relationship between generalized rough sets based on binary relation and covering,” *Inf. Sci.*, vol. 179, no. 3, pp. 210–225, 2009.

[15] Y. Y. Yao, “On generalizing Pawlak approximation operators,” in *Rough Sets and Current Trends in Computing*, ser. Lecture Notes in Computer Science, L. Polkowski and A. Skowron, Eds., vol. 1424. Springer, 1998, pp. 298–307.

[16] B. A. Davey and H. A. Priestly, *Introduction to Lattices and Order*. Cambridge: Cambridge University Press, 2002.

[17] Z. Csajbók, “Partial approximative set theory: A generalization of the rough set theory,” in *Proceedings of SoCPaR 2010, December 7-10, 2010., Cergy Pontoise / Paris, France*, T. Martin, A. K. Muda, A. Abraham, H. Prade, A. Laurent, D. Laurent, and V. Sans, Eds. IEEE, 2010, pp. 51–56.

[18] I. Düntsch and G. Gediga, “Approximation operators in qualitative data analysis,” in *Theory and Applications of Relational Structures as Knowledge Instruments*, ser. LNCS, H. C. M. de Swart, E. Orlowska, G. Schmidt, and M. Roubens, Eds., vol. 2929. Springer, 2003, pp. 214–230.

[19] P. Pagliani and M. Chakraborty, *A Geometry of Approximation: Rough Set Theory Logic, Algebra and Topology of Conceptual Patterns (Trends in Logic)*. Springer Publishing Company, Incorporated, 2008.

[20] D. Ciucci, “Approximation algebra and framework,” *Fundamenta Informaticae*, vol. 94, 2009, pp. 147–161.

[21] Z. Pawlak, *Rough Sets: Theoretical Aspects of Reasoning about Data*. Kluwer Academic Publishers, Dordrecht, 1991.

[22] A. Skowron, “Vague concepts: A rough-set approach,” in *Current Issues in Data and Knowledge Engineering: Proceedings of EUROFUSE 2004, The Eighth Meeting of the EURO Working Group on Fuzzy Sets, Workshop on Data and Knowledge Engineering, September 22-25, 2004, Warszawa, Poland*, B. De Baets, R. De Caluwe, G. De Tré, J. Fodor, J. Kacprzyk, and S. Zadrożny, Eds. Warszawa: Akademicka Oficyna Wydawnicza EXIT, 2004, pp. 480–493.

[23] Z. Csajbók, “Approximation of sets based on partial covering,” *Theoretical Computer Science*, vol. 412, no. 42, 2011, pp. 5820 – 5833.

[24] P. Odifreddi, *Classical Recursion Theory. The Theory of Functions and Sets of Natural Numbers*, ser. Studies in Logic and the Foundations of Mathematics. Elsevier, 1989, no. 125.

[25] A. Salomaa, *Computation and automata*, ser. Encyclopedia of Mathematics and its Applications. New York, NY, USA: Cambridge University Press, 1985, vol. 25.

Author Biographies

Zoltán Csajbók received his MSc degree in Mathematics from Kossuth University (Debrecen, Hungary) in 1979. Currently he is a senior research scientist at the Department of Health Informatics, Faculty of Health, University of Debrecen. Rough Set Theory, its generalizations and applications belong to his main scientific interest.

Tamás Mihálydeák has MSc degree in Mathematics (from Kossuth University), MA degree in Philosophy (from Eötvös University, Budapest, Hungary) and CSc degree (from Hungarian Academy of Science). He defended his habilitation thesis in 2008 at University of Debrecen. Presently he is an associate professor of the Department of Computer Science, Faculty of Informatics, University of Debrecen. His scientific interest covers non–classical logic, partial logic, intensional logic and their applications.