Partial Approximative Set Theory:  
A Generalization of the Rough Set Theory

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Abstract: The paper presents a generalization of the classical rough set theory, called the partial approximative set theory (PAST). According to Pawlak’s rough set theory, the vagueness of a subset of a finite universe $U$ is defined by the difference of its upper and lower approximations with respect to a $\sigma$-algebra generated by an equivalence relation on $U$. There are two most natural ways of the generalization of this idea. In particular, the equivalence relation is replaced by either any other type of binary relations on $U$ or an arbitrary covering of $U$. In this paper, our starting point will be a partial covering of an arbitrary universe. In general, the family of sets neither covers the universe nor forms a $\sigma$-algebra. We will put our discussions into an overall treatment called the general set theoretic approximation framework. We will investigate under what conditions our generalized upper and lower approximation pair forms Galois connection.

Keywords: vagueness, approximation of sets, rough set theory, partial approximative set theory, Galois-connection.

I. Introduction

The rough set theory (RST) was invented by the Polish mathematician, Zdzislaw Pawlak in the early 1980s [1, 2]. It can be seen as a new mathematical approach to manage uncertainty, incomplete, inexact or vague knowledge [3].

In its classical form, the starting point is a nonempty finite set $U$ of distinguishable objects, called the universe of discourse, and an equivalence relation $\varepsilon$ on $U$. The partition of $U$ generated by $\varepsilon$ is denoted by $U/\varepsilon$, and its elements are called $\varepsilon$-elementary sets. An $\varepsilon$-elementary set can be viewed as a set of indiscernible objects characterized by the same available information about them [4, 5]. In addition, any union of $\varepsilon$-elementary sets is referred to as a definable set.

Any subset $X \subseteq U$ can be naturally approximated by two sets called the lower and upper $\varepsilon$-approximations of $X$. The lower $\varepsilon$-approximation of $X$ is the union of all the $\varepsilon$-elementary sets which are the subsets of $X$, whereas the upper $\varepsilon$-approximation of $X$ is the union of all the $\varepsilon$-elementary sets that have a nonempty intersection with $X$.

The difference of upper and lower $\varepsilon$-approximations is called the $\varepsilon$-boundary of $X$. $X$ is exact ($\varepsilon$-crisp), if its $\varepsilon$-boundary is the empty set, inexact ($\varepsilon$-rough) otherwise.

Let $\sigma(U/\varepsilon) \subseteq 2^U$ denote the extension of $U/\varepsilon$ with all the unions of some $\varepsilon$-elementary sets and the empty set. $\sigma(U/\varepsilon)$ is a $\sigma$-algebra with the basis $U/\varepsilon$. In other words, $(U, \sigma(U/\varepsilon))$ is an Alexandrov topological space, where $\sigma(U/\varepsilon)$ is the family of all open and closed sets [6, 7].

In Pawlak’s theory, lower and upper $\varepsilon$-approximations can be defined in three equivalent forms. They are based on elements of $U$, $\varepsilon$-elementary sets and the $\sigma$-algebra $\sigma(U/\varepsilon)$ [8, 9, 10]. The generalization of Pawlak’s approximations can go along one of the three equivalent definitions.

The most natural generalization of Pawlak’s idea is that the equivalence relation is replaced by any other type of binary relation on $U$ [11, 6, 12]. Another way is that the partition is replaced by any covering of $U$ [13, 14]. The third way is to use two different subsystems of the powerset of $U$ [15]. A subsystem for the lower approximation which must be closed under unions and another for the upper approximation which, in turn, must be closed under intersections.

In this paper, our starting point will be a partial covering of an arbitrary universe $U$. The family of sets generally neither covers the universe nor forms a $\sigma$-algebra. We will put our discussions into an overall treatment called the general approximation framework. Within this framework, our concepts of lower and upper approximations are straightforward point-free generalizations of Pawlak’s ones. This new approach is called the partial approximative set theory (PAST).

The rest of the paper is organized as follows. In Section II we summarize the basic notations. Section III outlines two general approximation frameworks, a large-scaled one and a much finer one. This is one of the major contributions of our paper. Section IV presents the fundamental concepts and their properties of the classic Pawlak’s rough set theory. Only those facts will be considered which are important to the forthcoming ones. We provide new elementary point-free proofs for some of them. Section V sums up the basic principles of the partial approximative set theory. In Section VI we will investigate under what conditions our generalized upper and lower approximation pair forms Galois connection.
II. Basic Notations

Let $U$ be a nonempty set. Let $\mathcal{A} \subseteq 2^U$ be a family of subsets of $U$. The union of $\mathcal{A}$ is $\bigcup \mathcal{A} = \{x \mid \exists A \in \mathcal{A}(x \in A)\}$, and the intersection of $\mathcal{A}$ is $\bigcap \mathcal{A} = \{x \mid \forall A \in \mathcal{A}(x \in A)\}$. If $\mathcal{A}$ is an empty family of sets we define $\bigcup \emptyset = \emptyset$ and $\bigcap \emptyset = U$.

If $\epsilon$ is an arbitrary binary relation on $U$, let $[x]_\epsilon$ denote the $\epsilon$-related elements to $x$, i.e., $[x]_\epsilon = \{y \in U \mid (x, y) \in \epsilon\}$. The family of $[x]_\epsilon$ is denoted by $X/\epsilon$.

A nonempty set $P$ together with a partial order $\leq$ on $P$ is called a poset, in symbols $(P, \leq)$.

A self-map $f : P \to P$ is extensive if $x \leq f(x)$, contractive if $f(x) \leq x$.

If $(P, \leq P)$ and $(Q, \leq Q)$ are two posets, a map $f : P \to Q$ is monotone when $x \leq P y \Rightarrow f(x) \leq Q f(y)$, and antitone when $x \leq P y \Rightarrow f(y) \leq Q f(x)$, and order isomorphism if $f$ is a bijection and both $f$ and $f^{-1}$ are monotone.

Let $(P, \leq P)$ and $(Q, \leq Q)$ be two posets, and $(f, g)$ denote a pair of maps $f : P \to Q$, $g : Q \to P$. $(f, g)$ is a Galois connection between $P$ and $Q$ if

$$\forall p \in P \forall q \in Q \ (f(p) \leq Q q \Leftrightarrow p \leq P g(q)).$$

$f$ is called the lower adjoint and $g$ the upper adjoint of the Galois connection.

We also write $(P, f, g, Q)$ for a whole Galois connection. If $P = Q$ it is said $(P, f, g, P)$ is a Galois connection on $P$.

Remark 1. Here we adopted the definition of Galois connection in which the maps are monotone. It is also called the monotone or covariant form. For more details, see, e.g. [16].

The following proposition gives a useful characterization of Galois connections.

**Proposition 2** (E.g., [11], Lemma 79). The pair $(f, g)$ is a Galois connection if and only if

1. $\forall p \in P \ (p \leq P g(f(p)))$ and $\forall q \in Q \ (f(g(q)) \leq Q q)$; and
2. the maps $f$ and $g$ are monotone.

III. General Approximation Frameworks

In order to be able to treat the common features of both rough set theory and partial approximative theory uniformly, we define two general approximation frameworks, a large-scaled initial one, and a much finer general set theoretic one.

A. An Initial Approximation Framework

A large-scaled general framework of the set approximation with a pair of lower and upper approximation maps has been proposed in [17]. It is based on [18] and [19]. The framework has a specific prerequisite, in particular, the subsets of a set are approximated by the beforehand given subsets of the set itself.

Let $U$ be a nonempty set and $(l, u)$ be a pair of maps

$$l, u : 2^U \to 2^U.$$

The maps $l$ and $u$ are, of course, intended to be the lower and upper approximations of any subset $X \subseteq U$.

In this context, the nature of an approximation pair—beyond how they relate to one another—depends on how the lower and upper approximations of subsets relate to the subsets themselves.

The most essential features of approximation pairs $(l, u)$ of this type are specified as follows.

1. (**Monotonicity**) The maps $l, u : 2^U \to 2^U$ are monotone.

2. (**Weak approximation property.**) A pair of maps $(l, u)$ is the weak approximation pair on $U$ if they are monotone and

$$\forall X \in 2^U \ (l(X) \subseteq u(X)).$$

3. (**Strong approximation property.**) A pair of maps $(l, u)$ is the strong approximation pair on $U$, if each subset $X \subseteq 2^U$ is bounded by $l(X)$ and $u(X)$:

$$\forall X \in 2^U \ (l(X) \subseteq X \subseteq u(X)).$$

4. (**Approximation hypothesis.**) A pair of maps $(l, u)$ forms Galois connection on $(2^U, \subseteq)$ if

$$\forall X \in 2^U \forall Y \in 2^U \ (l(X) \subseteq Y \iff X \subseteq u(Y)).$$

Remark 3. Ad 1. This property is a common and reasonable assumption.

Ad 2. This constraint seems to be the weakest condition for a sensible concept of approximation of subsets in $U$ [20, 18].

Ad 4. In [19], a new hypothesis about approximation has been drawn up recently. According to this assumption, the notion of the “approximation” may be mathematically modelled by the notion of the Galois connection.

A much finer characterization of the nature of set approximations can be obtained with further specifications concerning the set families $l(Q^U)$ and $u(Q^U)$. These additional specifications will be performed in the next Subsection.

B. A General Set Theoretic Framework of Set Approximation

Let $U$ be an arbitrary nonempty set called the universe of discourse.

Let $\mathcal{D} \subseteq 2^U$ be a nonempty family of sets so that $\emptyset \in \mathcal{D}$ and it contains at least a nonempty subset $D \in 2^U$. The members of $\mathcal{D}$ are called definable sets, while the members of $2^U \setminus \mathcal{D}$ are undefinable.

We want to approximate of any subset $S \subseteq 2^U$ from “lower side” and “upper side”—no matter what they mean at this time. We have the only requirement at the highest level of abstraction that to let the lower and upper approximations of subsets $S$ be definable. We look at definable sets as tools to approximate subsets.

The following definition, at the next level of abstraction, is about the minimum requirements of lower and upper approximation mappings.

**Definition 4.** A pair $(l, u)$ of maps $l, u : 2^U \to \mathcal{D}$ is the weak approximation pair on $U$ if

(C1) $l$ and $u$ are monotone (monotonicity);

(C2) $u(\emptyset) = \emptyset$ (normality of $u$);

(C3) if $D \in \mathcal{D}$, then $l(D) = D$ (granularity of $\mathcal{D}$);

(C4) if $S \subseteq 2^U$, then $l(S) \subseteq u(S)$ (approximation property).
Clearly, the maps $l$, $u$ are total and many-to-one. According to the next proposition, $l$ is surjective, but $u$ in not necessarily surjective.

**Proposition 5.** Let $l, u : 2^U \to \mathcal{D}$ be a weak approximation pair on $U$.

1. $l(\emptyset) = \emptyset$ (normality of $l$);
2. $\forall X \in 2^U \ (l(l(X)) = l(X))$ (idempotency of $l$).
3. $S \in \mathcal{D}$ if and only if $l(S) = S$.
   In other words, $l(2^U) = \mathcal{D}$, i.e. $l$ is surjective.
4. $u(2^U) \subseteq l(2^U) = \mathcal{D}$.
   In other words, $u$ is not necessarily surjective.

**Proof.**

1. By definition, $\emptyset \in \mathcal{D}$ and so $l(\emptyset) = \emptyset$ by condition (C3).
2. $l(X) \in \mathcal{D}$ and so $l(l(X)) = l(X)$ by condition (C3).
3. $(\Rightarrow)$ It is just the same as the condition (C3).
   $(\Leftarrow)$ Since $l(S) \in \mathcal{D}$, and so $l(S) = S \in \mathcal{D}$.
4. Let $S \in u(2^U) \subseteq \mathcal{D}$. By the condition (C3), $S = l(S) \in l(2^U)$.

$\square$

The following example shows that each condition in Definition 4 is independent of the other three.

**Example 6.** Let $U$ be a nonempty set. Let us assume that there exist $B_1, B_2 (\not= \emptyset) \in 2^U$ so that neither $B_1 \subseteq B_2$ nor $B_2 \subseteq B_1$ holds, and there exists a proper superset $S$ of $B_1$ (i.e. $\emptyset \not= B_1 \subsetneq S \not= U$).

1. Let $\mathcal{D} = \{\emptyset, B_1, B_2, B_1 \cup B_2\}$ and $l, u : 2^U \to \mathcal{D}$ be as follows:
   
   \[
   l(X) = \begin{cases} 
   B_1, & \text{if } X = B_1; \\
   B_2, & \text{if } X = B_2; \\
   B_1 \cup B_2, & \text{if } X = B_1 \cup B_2, U; \\
   \emptyset, & \text{otherwise}.
   \end{cases}
   \]

   \[
   u(X) = \begin{cases} 
   \emptyset, & \text{if } X = \emptyset; \\
   B_1, & \text{if } X = B_1; \\
   B_1 \cup B_2, & \text{if } X = B_1 \cup B_2, U; \\
   B_2, & \text{otherwise}.
   \end{cases}
   \]

   Conditions (C2), (C3) trivially hold. Let us check the condition (C4):

   
   \[
   l(\emptyset) = \emptyset \subseteq \emptyset = u(\emptyset)
   \]

   
   \[
   l(B_1) = B_1 \subseteq B_1 = u(B_1)
   \]

   
   \[
   l(B_2) = B_2 \subseteq B_2 = u(B_2)
   \]

   
   \[
   l(B_1 \cup B_2) = B_1 \cup B_2 \subseteq B_1 \cup B_2 = u(B_1 \cup B_2)
   \]

   
   \[
   l(U) = B_1 \cup B_2 \subseteq B_1 \cup B_2 = u(U)
   \]

   
   \[
   l(S) = \emptyset \subseteq B_2 = u(S)
   \]

   and if $S' (\not= \emptyset, B_1, B_2, B_1 \cup B_2, S, U) \in 2^U$, then

   
   \[
   l(S') = \emptyset \subseteq B_2 = u(S').
   \]

   That is the condition (C4) also holds. However, in the case $B_1 \subsetneq S$

   
   \[
   l(B_1) = B_1 \not\subseteq \emptyset = l(S)
   \]

   
   \[
   u(B_1) = B_1 \not\subseteq B_2 = u(S)
   \]

   Therefore, these $l$ and $u$ satisfy all the four conditions except (C1).

2. Let $\mathcal{D} = \{\emptyset, B_1, B_2, B_1 \cup B_2\}$ and $l, u : 2^U \to \mathcal{D}$ be as follows:

   \[
   l(X) = \begin{cases} 
   \emptyset, & \text{if } X = \emptyset; \\
   B_1, & \text{if } X = B_1; \\
   B_2, & \text{if } X = B_2; \\
   B_1 \cup B_2, & \text{otherwise}.
   \end{cases}
   \]

   \[
   u(X) = \begin{cases} 
   \emptyset, & \text{if } X = \emptyset; \\
   B_1, & \text{if } X = B_1; \\
   B_1 \cup B_2, & \text{otherwise}.
   \end{cases}
   \]

   Conditions (C1), (C3), (C4) hold, but $u(\emptyset) = B_1 \cup B_2$.

   Therefore, these $l$ and $u$ satisfy all the four conditions except (C2).

3. Let $\mathcal{D} = \{\emptyset, B_2, B_1 \cup B_2\}$ and $l, u : 2^U \to \mathcal{D}$ be as follows:

   \[
   l(X) = \begin{cases} 
   \emptyset, & \text{if } X = \emptyset; \\
   B_1, & \text{if } X = B_1; \\
   B_1 \cup B_2, & \text{otherwise}.
   \end{cases}
   \]

   \[
   u(X) = \begin{cases} 
   \emptyset, & \text{if } X = \emptyset; \\
   B_1 \cup B_2, & \text{otherwise}.
   \end{cases}
   \]

   Conditions (C1), (C2) trivially hold. Let us check the condition (C4):

   \[
   l(\emptyset) = \emptyset \subseteq \emptyset = u(\emptyset)
   \]

   \[
   l(B_1) = B_1 \subseteq B_1 = u(B_1)
   \]

   \[
   l(B_2) = B_2 \subseteq B_2 = u(B_2)
   \]

   \[
   l(B_1 \cup B_2) = B_1 \cup B_2 \subseteq B_1 \cup B_2 = u(B_1 \cup B_2)
   \]

   \[
   l(U) = B_1 \cup B_2 \subseteq B_1 \cup B_2 = u(U)
   \]

   \[
   l(S) = \emptyset \subseteq B_2 = u(S)
   \]

   and if $S' (\not= \emptyset, B_1, B_2, B_1 \cup B_2, S, U) \in 2^U$, then

   \[
   l(S') = \emptyset \subseteq B_2 = u(S').
   \]

   That is the condition (C4) also holds. However, if $\emptyset \not= B_1 \subsetneq S$

   \[
   l(B_1) = B_1 \not\subseteq \emptyset = l(S)
   \]

   \[
   u(B_1) = B_1 \not\subseteq B_2 = u(S)
   \]

   Therefore, these $l$ and $u$ satisfy all the four conditions except (C3).

4. Let $\mathcal{D} = \{\emptyset, B_1, B_2, B_1 \cup B_2\}$ and $l, u : 2^U \to \mathcal{D}$ be as follows:

   \[
   l(X) = \begin{cases} 
   \emptyset, & \text{if } X = \emptyset; \\
   B_1, & \text{if } X = B_1; \\
   B_2, & \text{if } X = B_2;
   \end{cases}
   \]

   \[
   u(X) = \begin{cases} 
   \emptyset, & \text{if } X = \emptyset; \\
   B_1 \cup B_2, & \text{otherwise}.
   \end{cases}
   \]

   These $l$ and $u$ trivially satisfy all the four conditions except (C4).

Next definition classifies the lower and upper approximation pairs as how the lower and upper approximations of a subset relate to the subset itself.

**Definition 7.** A pair $(l, u)$ of maps $l, u : 2^U \to \mathcal{D}$ is

- the $l$-semi-strong approximation pair on $U$ if it is weak and if $S \subseteq 2^U$, then $l(S) \subseteq S \subseteq (l(S))$ (l is contractive);
- the $u$-semi-strong approximation pair on $U$ if it is weak and if $S \subseteq 2^U$, then $S \subseteq u(S)$ (l is extensive);
- the strong approximation pair on $U$ if it is $l$-semi-strong and $u$-semi-strong at the same time, i.e. each subset $S \subseteq 2^U$ is bounded by $l(X)$ and $u(X)$: $\forall S \subseteq 2^U \ (l(S) \subseteq S \subseteq u(X))$. 


If \( U \) is a nonempty set, and \( \mathcal{D} = 2^U \), it is straightforward that the pair of maps \( l, u : 2^U \to \mathcal{D}, X \mapsto X \) is a strong approximation pair.

The next example shows that there are approximation pairs which are neither \( l \)-semi-strong nor \( u \)-semi-strong, not \( l \)-semi-strong but \( u \)-semi-strong, \( l \)-semi-strong but not \( u \)-semi-strong.

Example 8. Let \( U = \{a, b\} \) be a nonempty set.

1. Let \( \mathcal{D} = \{\emptyset, \{a\}\} \), and the maps \( l, u : 2^U \to \mathcal{D} \) be as follows:
   \[
   X \mapsto l(X), u(X) = \begin{cases} 
   \emptyset, & \text{if } X = \emptyset; \\
   \{a\}, & \text{otherwise}. 
   \end{cases}
   \]
   Conditions (C1)–(C4) can easily be checked. However, for \( X = \{\emptyset\} \)
   \[
   l(\{\emptyset\}) = \{a\} \not\subseteq \{\emptyset\} \not\subseteq \{a\} = u(\{\emptyset\}).
   \]
   Therefore, the approximation pair \( \langle l, u \rangle \) is neither \( l \)-semi-strong nor \( u \)-semi-strong.

2. Let \( \mathcal{D} = \{\emptyset, \{a\}, \{a, b\}\} \), and \( l, u : 2^U \to \mathcal{D} \) be as follows:
   \[
   X \mapsto l(X), u(X) = \begin{cases} 
   \emptyset, & \text{if } X = \emptyset; \\
   \{a\}, & \text{if } X = \{a\}; \\
   \{a, b\}, & \text{otherwise}. 
   \end{cases}
   \]
   Conditions (C1)–(C4) can easily be checked. Let us check that \( u \) is extensive:
   - \( \emptyset \subseteq \emptyset = u(\emptyset) \);
   - \( \{a\} \subseteq \{a\} = u(\{a\}) \);
   - \( \{b\} \subseteq \{a, b\} = u(\{b\}) \);
   - \( \{a, b\} \subseteq \{a, b\} = u(\{a, b\}) \).
   However, in the case \( X = \{b\} \),
   \[
   l(\{b\}) = \{a, b\} \not\subseteq \{b\} \not\subseteq \{a, b\} = u(\{b\}).
   \]
   Therefore, the approximation pair \( \langle l, u \rangle \) is not \( l \)-semi-strong, but \( u \)-semi-strong.

3. Let \( \mathcal{D} = \{\emptyset, \{a\}, \{b\}\} \), and \( l, u : 2^U \to \mathcal{D} \) be as follows:
   \[
   X \mapsto l(X), u(X) = \begin{cases} 
   \emptyset, & \text{if } X = \emptyset; \\
   \{a\}, & \text{if } X = \{a\}; \\
   \{b\}, & \text{otherwise}. 
   \end{cases}
   \]
   Conditions (C1)–(C4) can easily be checked. Let us check that \( l \) is contractive:
   - \( l(\emptyset) = \emptyset \subseteq \emptyset \);
   - \( l(\{a\}) = \{a\} \subseteq \{a\} \);
   - \( l(\{b\}) = \{b\} \subseteq \{b\} \);
   - \( l(\{a, b\}) = \{a, b\} \subseteq \{a, b\} \).
   However, in the case \( X = \{a, b\} \),
   \[
   l(\{a, b\}) = \{a, b\} \not\subseteq \{a, b\} \not\subseteq \{a, b\} = u(\{a, b\}).
   \]
   Therefore, the approximation pair \( \langle l, u \rangle \) is \( l \)-semi-strong, but not \( u \)-semi-strong.

Using the preliminary notations general approximation spaces can be defined.

Definition 9. An ordered quadruple \( \langle U, \mathcal{D}, l, u \rangle \) is the weak/l-/semi-strong/u-/semi-strong/strong generalized approximation space, if the approximation pair \( \langle l, u \rangle \) is weak/l-/semi-strong/u-/semi-strong/strong, respectively.

Proposition 10. Let \( \langle U, \mathcal{D}, l, u \rangle \) be a generalized approximation space.

1. If \( \langle U, \mathcal{D}, l, u \rangle \) is weak, then
   - (a) \( l(U) \subseteq \bigcup \mathcal{D} \)
   - (b) \( l(U) = \bigcup \mathcal{D} \) if and only if \( \bigcup \mathcal{D} \in \mathcal{D} \).
   - (c) \( u(U) \subseteq \bigcup \mathcal{D} \).

Proof. 1. (a) By the definition of \( l \), \( l(U) \in \mathcal{D} \) and so \( l(U) \subseteq \bigcup \mathcal{D} \).
   - (b) \( (\Rightarrow) \) By the definition of \( l \), \( l(U) = \bigcup \mathcal{D} \in \mathcal{D} \).
   - \( (\Leftarrow) \) Let us assume that \( \bigcup \mathcal{D} \in \mathcal{D} \). Since \( \bigcup \mathcal{D} \subseteq U \), then by the condition (C3) and the monotonicity of \( l \), \( l(\bigcup \mathcal{D}) = \bigcup \mathcal{D} \subseteq l(U) \). Comparing it with (1) (a), we obtain \( l(U) = \bigcup \mathcal{D} \).
   - (c) By the definitions of \( u \), \( u(U) \in \mathcal{D} \) and so \( u(U) \subseteq \bigcup \mathcal{D} \).

2. \( \langle U, \mathcal{D}, l, u \rangle \) is weak, thus by Proposition 10 (1)/(c), \( u(U) \subseteq \bigcup \mathcal{D} \). On the other hand, since \( u \) is monotone and extensive, \( \bigcup \mathcal{D} \subseteq U \) implies \( \bigcup \mathcal{D} \subseteq u(U) \). Consequently, \( u(U) = \bigcup \mathcal{D} \).

Clearly, \( u(U) \subseteq U \). Since \( u \) is extensive, thus \( U \subseteq u(U) \). Therefore, \( u(U) = U \).

In generalized approximation spaces the notion of well approximated sets can be introduced which we call crisp sets.

Definition 11. Let \( \langle U, \mathcal{D}, l, u \rangle \) be a weak/l-/semi-strong/u-/semi-strong/strong generalized approximation space and \( S \in 2^U \).

\( S \) is a weak/l-/semi-strong/u-/semi-strong/strong crisp set with respect to the given weak/l-/semi-strong/u-/semi-strong/strong generalized approximation space, if \( l(S) = u(S) \).

Proposition 12. Let \( \langle U, \mathcal{D}, l, u \rangle \) be a strong generalized approximation space.

If \( S \in 2^U \) is a strong crisp set, then \( S \) is definable.

Proof. In the strong generalized approximations space \( \langle U, \mathcal{D}, l, u \rangle \), \( l(S) \subseteq S \subseteq u(S) \). Since \( S \) is crisp, therefore \( l(S) = S = u(S) \), and so \( S \in \mathcal{D} \) by Proposition 5, point 3.
IV. Basics of Rough Set Theory

The basic concepts and properties of rough set theory can be found, e.g., in [21], [11]. Here we cite only a few of them which will be important in the following. We provide new elementary point-free proofs for some of them.

Definition 13. A pair \((U, \varepsilon)\), where \(U\) is a finite universe of discourse and \(\varepsilon\) is an equivalence relation on \(U\), is called Pawlak’s approximation space.

A subset \(X \subseteq U\) is \(\varepsilon\)-definable, if it is a union of \(\varepsilon\)-elementary sets, otherwise \(X\) is \(\varepsilon\)-undefinable.

By definition, the empty set is considered to be an \(\varepsilon\)-definable set.

Let \(\mathcal{D}_{U/\varepsilon}\) denote the family of \(\varepsilon\)-definable subsets of \(U\).

Remark 14. For an evolutionary survey of approximation spaces, see [22].

The following lemma is elementary, however, in the context of Pawlak’s rough set theory it is an important fact. It follows from just the fact that the partition \(U/\varepsilon\) consists of nonempty pairwise disjoint subsets of \(U\).

Lemma 15. \(\forall X \in 2^{U/\varepsilon} \forall X \in U/\varepsilon (X \subseteq U \Leftrightarrow X \in \mathfrak{X})\).

Proposition 16. Let \((U, \varepsilon)\) be Pawlak’s approximation space.

Then \((2^{U/\varepsilon}, \subseteq)\) and \((\mathcal{D}_{U/\varepsilon}, \subseteq)\) are order isomorphic via the map \(u_\varepsilon : 2^{U/\varepsilon} \rightarrow \mathcal{D}_{U/\varepsilon}, \mathfrak{X} \mapsto \mathfrak{X}\).

Proof. We show that the map \(u_\varepsilon\) is a bijection and both \(u_\varepsilon\) and \(u_\varepsilon^{-1}\) are monotone.

Let \(D_1, D_2 \in 2^{U/\varepsilon}\) be such that \(\bigcup \mathfrak{X}_1 = \bigcup \mathfrak{X}_2 \in \mathcal{D}_{U/\varepsilon}\).

By Lemma 15, \(\forall X \in 2^{U} (X \in \mathfrak{X}_1 \Leftrightarrow X \subseteq \bigcup \mathfrak{X}_1 \Leftrightarrow X \subseteq \bigcup \mathfrak{X}_2 \Leftrightarrow X \in \mathfrak{X}_2)\), i.e., \(\mathfrak{X}_1 \subseteq \mathfrak{X}_2\), and so \(u_\varepsilon^{-1}\) is also monotone.

In Pawlak’s approximation spaces, the lower and upper approximations of \(X\) can be defined in the point-free manner—based on the \(\varepsilon\)-elementary sets, and in the point-wise manner—based on the elements.

Definition 17. Let \((U, \varepsilon)\) be Pawlak’s approximation space, and \(X \in 2^{U}\) be a subset of \(U\).

The lower \(\varepsilon\)-approximation of \(X\) is

\[
\underline{X} = \bigcup \{Y \mid Y \in U/\varepsilon, Y \subseteq X\} = \{x \in U \mid \text{[x]}_{\varepsilon} \subseteq X\},
\]

and the upper \(\varepsilon\)-approximation of \(X\) is

\[
\overline{X} = \bigcup \{Y \mid Y \in U/\varepsilon, Y \cap X \neq \emptyset\} = \{x \in U \mid \text{[x]}_{\varepsilon} \cap X \neq \emptyset\}.
\]

The set \(B_{\varepsilon}(X) = \overline{X} \setminus \underline{X}\) is the \(\varepsilon\)-boundary of \(X\). \(X\) is \(\varepsilon\)-crisp, if \(B_{\varepsilon}(X) = \emptyset\), otherwise \(X\) is \(\varepsilon\)-rough.

It follows from just the definitions that \(\underline{X}, \overline{X} \in \mathcal{D}_{U/\varepsilon}\), the maps \(\underline{\cdot}, \overline{\cdot} : 2^{U} \rightarrow \mathcal{D}_{U/\varepsilon}\) are total, onto and many-to-one.

Proposition 18 ([21], Proposition 2.1, point a). Let \((U, \varepsilon)\) be Pawlak’s approximation space. Then \(X \in \mathcal{D}_{U/\varepsilon}\) if and only if \(\underline{X} = \overline{X}\).

Proposition 19 ([21], Proposition 2.2, points 1). Let \((U, \varepsilon)\) be Pawlak’s approximation space. Then

\[
\forall X \in 2^{U} (\underline{X} \subseteq X \subseteq \overline{X}),
\]

that is, the maps \(\underline{\cdot}\) and \(\overline{\cdot}\) are contractive and extensive, respectively.

In other words, the pair of maps \(\underline{\cdot}\) and \(\overline{\cdot}\) is a strong approximation pair on \(U\).

Corollary 20. \(\underline{X} = X\) if and only if \(X = \overline{X}\).

Proof. Since \(\underline{X} \in \mathcal{D}_{U/\varepsilon}\), then \(X = \underline{X}\). Therefore, \(\underline{X} \subseteq X\). By Proposition 19, \(X = \underline{X}\). Thus, \(\underline{X} = X\).

Proposition 21. Let \((U, \varepsilon)\) be Pawlak’s approximation space and \(X \subseteq U\).

1. \(X\) is \(\varepsilon\)-crisp if and only if \(X\) is \(\varepsilon\)-definable.

2. \(X\) is \(\varepsilon\)-rough if and only if \(X\) is \(\varepsilon\)-undefinable.

Proof.

1. \((\Rightarrow)\) \(X\) is \(\varepsilon\)-crisp \(\Rightarrow\) \(B_{\varepsilon}(X) = \overline{X} \setminus \underline{X} = \emptyset \Rightarrow \underline{X} = X\). By definition, \(\underline{X} \subseteq X\). Proposition 19 implies \(\underline{X} \subseteq \overline{X}\), and so \(\underline{X} = X\). According to Proposition 18, \(\underline{X} = X\).

2. \((\Leftarrow)\) Since \(X \in \mathcal{D}_{U/\varepsilon}\), \(\underline{X} = X\), so \(B_{\varepsilon}(X) = \emptyset\). Thus, \(\emptyset = \emptyset\).

It is the contrapositive version of 1.

As a consequence of Proposition 21, the notions ‘\(\varepsilon\)-crisp’ and ‘\(\varepsilon\)-definable’ are synonymous to each other, and so are ‘\(\varepsilon\)-rough’ and ‘\(\varepsilon\)-undefinable’.

Lower and upper \(\varepsilon\)-approximations can be generalized via their point-wise definitions [11].

Definition 22. Let \(\varepsilon\) be an arbitrary binary relation on \(U\) and \(X \subseteq 2^{U}\). The lower \(\varepsilon\)-approximation of \(X\) is

\[
\underline{X} = \{x \in U \mid [x]_{\varepsilon} \subseteq X\},
\]

and the upper \(\varepsilon\)-approximation of \(X\) is

\[
\overline{X} = \{x \in U \mid [x]_{\varepsilon} \cap X \neq \emptyset\}.
\]

If \(\varepsilon^{-1}\) denotes the inverse relation of \(\varepsilon\), in the same manner one can also define lower and upper \(\varepsilon^{-1}\)-approximations.

Proposition 23 ([11], Proposition 134). Let \(\varepsilon\) be an arbitrary binary relation on \(U\).

Then \(\langle 2^{U}, \varepsilon \setminus \varepsilon^{-1}, 2^{U} \rangle\) and \(\langle 2^{U}, \varepsilon^{-1} \setminus \varepsilon, 2^{U} \rangle\) are Galois connections on \(\langle 2^{U}, \subseteq \rangle\).

Some properties of lower and upper \(\varepsilon\)-approximations are expressed by properties of binary relations and vice versa.

Proposition 24. Let \(\varepsilon\) be an arbitrary binary relation on \(U\).

1. The pair \((\varepsilon, \varepsilon^{-1})\) is a weak approximation pair if and only if \(\varepsilon\) is connected.

2. The pair \((\varepsilon, \varepsilon^{-1})\) is a strong approximation pair if and only if \(\varepsilon\) is reflexive.
3. The pair \((\tau, \xi)\) is a Galois connection on \((2^U, \subseteq)\) if and only if \(\tau\) is symmetric.

In particular, if \(\varepsilon\) is an equivalence relation on \(U\), then \((2^U, \tau, \varepsilon, 2^U)\) is a Galois connection on \(2^U\).


It can be shown that even if the relation \(\varepsilon\) is symmetric, it is not sufficient that the lower and upper \(\varepsilon\)-approximations defined in the point-free manner form a Galois connection ([23], Example 3.10).

V. Partial Approximation of Sets

In practice, there are attributes which do not characterize all members of an observed collection of objects.

Some illustrative examples:

- With the property ‘color of hair’ bald men cannot be characterized.
- An infinite set is investigated via a finite family of its finite subsets. For instance, a number theorist studies the regularities of natural numbers using computers.
- Security policies are partial-natured in corporate information security management. Typically some policies may only apply to specific hardware appliances, software applications or type of information.

Moreover, there are some features with which a set and its complement cannot be considered at the same time. For instance, complements of recursively enumerable sets are not necessarily recursively enumerable. The membership of recursively enumerable sets can effectively be determined by a finite amount of information, while the determination of their nonmembership requires an infinite amount of information [24]. That is, the complement of a recursively enumerable set cannot necessarily be determined effectively. In other words, the recursively enumerable sets can be managed by computers (e.g., via a special rewriting system, the Markov algorithm [25]). Thus, this is an important practical partial approximation problem: how can we approximate an arbitrary set with recursively enumerable sets?

Throughout this section let \(U\) be a nonempty set called the universe of discourse.

A. Definable Sets

The first definition gives us the fundamental sets of our framework which can be considered as our primary tools.

Definition 25. Let \(\mathfrak{B} = \{B_i \mid i \in I\} \subseteq 2^U\) be a nonempty family of nonempty subsets of \(U\), where \(I\) denotes an index set.

\(\mathfrak{B}\) is called the base system, its elements are the \(\mathfrak{B}\)-sets.

Some extensions of the base set \(\mathfrak{B}\) can be defined by means of \(\mathfrak{B}\). It can be seen as derived tools. The next definition is about a possible extension of \(\mathfrak{B}\).

Definition 26. A set family \(\mathfrak{S} \subseteq 2^U\) is \(\mathfrak{B}\)-definable if its elements are \(\mathfrak{B}\)-sets, otherwise \(\mathfrak{S}\) is \(\mathfrak{B}\)-undefinite.

A nonempty subset \(S \subseteq 2^U\) is \(\mathfrak{B}\)-definable if there exists a \(\mathfrak{B}\)-definable set family \(\mathfrak{S} = \{B_i \mid \text{for all } i \in I, B_i \in \mathfrak{B}\}\) so that \(S = \bigcup \mathfrak{S}\), otherwise \(S\) is \(\mathfrak{B}\)-undefinite.

The empty set is considered to be a \(\mathfrak{B}\)-definable set.

Let \(\mathfrak{D}_\mathfrak{B}\) denote the family of \(\mathfrak{B}\)-definable subsets of \(U\).

Notice that \(\emptyset \in \mathfrak{D}_\mathfrak{B}\) and \(\mathfrak{B} \subseteq \mathfrak{D}_\mathfrak{B}\), therefore \(\mathfrak{D}_\mathfrak{B}\) contains at least a nonempty subset of \(U\) and \(\bigcup \mathfrak{B} = \bigcup \mathfrak{D}_\mathfrak{B} = \mathfrak{D}_\mathfrak{B}\). It is straightforward that \(\mathfrak{D}_\mathfrak{B}\) does not form a \(\sigma\)-algebra at all.

We will need the following notion.

Definition 27. The base system \(\mathfrak{B} \subseteq 2^U\) is single-layered, if

\[\forall B \in \mathfrak{B} \quad \forall \mathfrak{B}' \subseteq \mathfrak{B} \setminus \{B\} \quad (B \cap \bigcup \mathfrak{B}' \neq B).\]

Informally, a base system \(\mathfrak{B}\) is single-layered if every primary and derived tools has at least one element which can be characterized by exactly one primary tool.

Some properties of rough set theory can be preserved in some wise with the notion of single-layered. For instance:

Proposition 28 ([23], Proposition 4.5; analogous with Proposition 16). Let \(\mathfrak{B} \subseteq 2^U\) be a base system. Then \((2^\mathfrak{B}, \subseteq)\) and \((\mathfrak{D}_\mathfrak{B}, \subseteq)\) are order isomorphic via the map \(u_{\mathfrak{B}} : 2^\mathfrak{B} \rightarrow \mathfrak{D}_\mathfrak{B}, X \mapsto \bigcup X\) if and only if \(\mathfrak{B}\) is single-layered.

B. Lower and Upper \(\mathfrak{B}\)-Approximations

Definition 29. Let \(\mathfrak{B} \subseteq 2^U\) be a base system and \(X\) be a subset of \(U\).

The lower \(\mathfrak{B}\)-approximation of \(X\) is

\[c^\downarrow_\mathfrak{B}(X) = \bigcup \{Y \mid Y \in \mathfrak{B}, Y \subseteq X\},\]

and the upper \(\mathfrak{B}\)-approximation of \(X\) is

\[c^\uparrow_\mathfrak{B}(X) = \bigcup \{Y \mid Y \in \mathfrak{B}, Y \cap X \neq \emptyset\}.\]

Notice that \(c^\downarrow_\mathfrak{B}\) and \(c^\uparrow_\mathfrak{B}\) are straightforward point-free generalizations of Pawlak’s lower and upper \(\varepsilon\)-approximations. Clearly, \(c^\downarrow_\mathfrak{B}(X), c^\uparrow_\mathfrak{B}(X) \in \mathfrak{D}_\mathfrak{B}\), and the maps \(c^\downarrow_\mathfrak{B}, c^\uparrow_\mathfrak{B} : 2^U \rightarrow \mathfrak{D}_\mathfrak{B}\) are total, onto and many-to-one.

Proposition 30. Let \(\mathfrak{B} \subseteq 2^U\) be a base system.

1. \(\forall S \in 2^U\) \((c^\downarrow_\mathfrak{B}(S) \subseteq S)\)—that is \(c^\downarrow_\mathfrak{B}\) is contractive.

2. \(\forall S \in 2^U\) \((S \subseteq c^\downarrow_\mathfrak{B}(S))\) if and only if \(\bigcup \mathfrak{B} = U\)—that is \(c^\downarrow_\mathfrak{B}\) is extensive if and only if \(\mathfrak{B}\) covers the universe.

Proof. 1. is straightforward.

2. \((\Rightarrow)\) \(U \subseteq c^\downarrow_\mathfrak{B}(U) = \bigcup \{B \mid B \in \mathfrak{B}, B \subseteq U\} = \bigcup \mathfrak{B}\). Of course, \(\bigcup \mathfrak{B} \subseteq U\), and so \(\bigcup \mathfrak{B} = U\).

\((\Leftarrow)\) \(\forall S \in 2^U\) \((S \subseteq U = \bigcup \mathfrak{B})\), thus we get

\[S \subseteq \bigcup \mathfrak{B} \setminus \{B \mid B \in \mathfrak{B}, B \cap S = \emptyset\}\]

\[= \bigcup \{B \mid B \in \mathfrak{B}, B \cap S \neq \emptyset\} = c^\downarrow_\mathfrak{B}(S).\]

Proposition 31. Let \(\mathfrak{B} \subseteq 2^U\) be a base system.

1. The maps \(c^\downarrow_\mathfrak{B}, c^\uparrow_\mathfrak{B} : 2^U \rightarrow \mathfrak{D}_\mathfrak{B}\) are monotone.

2. \(c^\downarrow_\mathfrak{B}(\emptyset) = \emptyset\).

3. If \(D \in \mathfrak{D}_\mathfrak{B}\), then \(c^\downarrow_\mathfrak{B}(D) = D\).

4. If \(S \in 2^U\), then \(c^\downarrow_\mathfrak{B}(S) \subseteq c^\downarrow_\mathfrak{B}(S)\).
Proof. 1., 2. and 4. are straightforward by the definition of lower and upper \( B \)-approximations.

3. Clearly, if \( \emptyset \in \mathcal{D}_B \), then \( C^\sharp_B(\emptyset) = \emptyset \).

If \( \emptyset \notin \mathcal{D}_B \), there exists at least one nonempty family of sets \( \mathcal{B'} \subseteq \mathcal{B} \) so that \( D = \{ \mathcal{B'} = \bigcup \{ B \mid B \in \mathcal{B'}, B \subseteq D \} \subseteq \bigcup \{ B \mid B \in \mathcal{B}, B \subseteq D \} = C^\flat_B(D) \). On the other hand, we have \( C^\sharp_B(D) \subseteq D \). Thus \( C^\sharp_B(D) = D \). \( \square \)

In the language of general set theoretic framework of set approximation the pair \( (C^\sharp_B, C^\flat_B) \) of maps \( C^\sharp_B, C^\flat_B : \mathbb{2}^U \to \mathcal{D}_B \) is an l-semi-strong approximation pair, and it is a strong one if and only if the base system \( \mathcal{B} \) covers the universe.

Moreover, the so-called \( \mathcal{B} \)-approximation space \( (\mathcal{U}, \mathcal{C}^\sharp_B, \mathcal{C}^\flat_B) \) is l-semi-strong and it is strong if and only if the base system \( \mathcal{B} \) covers the universe.

VI. Galois Connections in \( \mathcal{B} \)-approximation Spaces

Let us investigate what conditions have to be satisfied by the l-semi-strong \( \mathcal{B} \)-approximation space \( (\mathcal{U}, \mathcal{B}, C^\sharp_B, C^\flat_B) \) so that the pair \( (C^\sharp_B, C^\flat_B) \) forms a Galois connection on \( U \).

The next proposition answers the first half of the Point 1 in Proposition 2.

**Proposition 32.** Let \( (U, \mathcal{B}, C^\sharp_B, C^\flat_B) \) be an l-semi-strong \( \mathcal{B} \)-approximation space.

Then \( \forall X \in 2^U (X \subseteq C^\flat_B (C^\sharp_B (X))) \) if and only if \( \bigcup \mathcal{B} = U \).

**Proof.** \((\Rightarrow)\) By a contradiction, let us assume that \( \bigcup \mathcal{B} \neq U \). Accordingly, \( \exists X' (\neq \emptyset) \subseteq U \setminus \bigcup \mathcal{B} \). Hence, \( C^\flat_B (C^\sharp_B (X')) = \emptyset \), which gives \( \emptyset \neq X' \subseteq C^\flat_B (C^\sharp_B (X')) = \emptyset \), a contradiction.

\((\Leftarrow)\) \( C^\flat_B (X) \in \mathcal{D}_B \), and so, by Proposition 31 Point 3, \( C^\flat_B (C^\sharp_B (X)) = C^\flat_B (X) \). Since \( \bigcup \mathcal{B} = U \), by Proposition 30 Point 2, \( C^\flat_B \) is extensive, thus \( X \subseteq C^\flat_B (X) = C^\flat_B (C^\sharp_B (X)) \). \( \square \)

Let us take up the question of the second half of the Point 1 in Proposition 2. In general, it also does not hold.

**Proposition 33.** Let \( (U, \mathcal{B}, C^\sharp_B, C^\flat_B) \) be an l-semi-strong \( \mathcal{B} \)-approximation space.

Then the base system \( \mathcal{B} \) is single-layered and \( \forall X \in 2^U (C^\flat_B (C^\sharp_B (X)) \subseteq X) \) if and only if the \( \mathcal{B} \)-sets are pairwise disjoint.

**Proof.** \((\Rightarrow)\) By a contradiction, let us assume that the \( \mathcal{B} \)-sets are not pairwise disjoint, i.e.

\[ \exists B_1, B_2 \in \mathcal{B} (B_1 \neq B_2 \land B_1 \cap B_2 \neq \emptyset) \]\

Because \( \mathcal{B} \) is single-layered, neither \( B_1 \subseteq B_2 \) nor \( B_2 \subseteq B_1 \) holds. Hence, e.g. for \( B_1 \), we have

\[ C^\flat_B (C^\sharp_B (B_1)) = C^\flat_B (B_1) = \bigcup \{ Y \mid Y \in \mathcal{B}, Y \cap B_1 \neq \emptyset \} \]\

Thus we get \( B_1 \cup B_2 \subseteq C^\flat_B (C^\sharp_B (B_1)), \) and so \( C^\flat_B (C^\sharp_B (B_1)) \nsubseteq B_1 \), a contradiction.

\((\Leftarrow)\) Clearly, if the \( \mathcal{B} \)-sets are pairwise disjoint, \( \mathcal{B} \) is single-layered. Furthermore, \( C^\flat_B (C^\sharp_B (\emptyset)) = C^\flat_B (\emptyset) = \emptyset \subseteq \emptyset \) holds, independently of that the \( \mathcal{B} \)-sets are pairwise disjoint or not.

Let \( 0 \neq X \in 2^U \). If \( C^\flat_B (X) = \emptyset \), then \( C^\flat_B (\emptyset) = \emptyset \subseteq X \).

Let \( C^\flat_B (X) = \bigcup B' \neq \emptyset \) for a family of \( \mathcal{B} \)-sets \( B' \subseteq \mathcal{B} \).

Because the map \( C^\flat_B \) is contractive, \( C^\flat_B (X) = \bigcup B' \subseteq X \).

Since the \( \mathcal{B} \)-sets are pairwise disjoint, \( \{ Y \mid Y \in \mathcal{B}, Y \cap \bigcup B' \neq \emptyset \} = \{ Y \mid Y \in \mathcal{B}, Y \cap \bigcup B' \neq \emptyset \} \). Thus we get

\[ C^\flat_B (C^\sharp_B (X)) = C^\flat_B (\bigcup B') = \bigcup \{ Y \mid Y \in \mathcal{B}, Y \cap \bigcup B' \neq \emptyset \} = \bigcup \{ Y \mid Y \in \mathcal{B}, Y \subseteq \bigcup B' \} \subseteq \bigcup \{ Y \mid Y \in \mathcal{B}, Y \subseteq X \} \subseteq X \]. \( \square \)

**Theorem 34.** Let \( (U, \mathcal{B}, C^\sharp_B, C^\flat_B) \) be an l-semi-strong \( \mathcal{B} \)-approximation space.

The base system \( \mathcal{B} \) is single-layered and the pair \( (C^\sharp_B, C^\flat_B) \) forms a Galois connection on \( (2^U, \subseteq) \) if and only if the base system \( \mathcal{B} \) is a partition of \( U \).

**Proof.** The maps \( C^\sharp_B \) and \( C^\flat_B \) are monotone, and so by Propositions 32 and 33, the conditions in Proposition 2 hold.

If the base system \( \mathcal{B} \) is a partition of \( U \), then \( \mathcal{B} \) is trivially single-layered.

However, if the \( (C^\sharp_B, C^\flat_B) \) forms a Galois connection in an l-semi-strong \( \mathcal{B} \)-approximation space, it can be proven that \( \mathcal{B} \) is a partition of \( U \) without the assumption that \( \mathcal{B} \) is single-layered. For details see [23], Theorem 4.14.

VII. Conclusion

In this paper, first, we have presented a general set theoretic approximation framework. Within this framework, a particular partial approximation space has been proposed. In Pawlak’s space upper and lower approximations form a Galois connection on \( U \). We have investigated what conditions have to be satisfied by our generalized upper and lower approximation pair forms a Galois connection on \( U \).

References

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